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DEVELOPMENT OF A SCIENTIFIC BASIS FOR ANALYSIS OF AIRCRAFT SEATING SYSTEMS

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16. Abstract
A three-dimensional mathematical model of an aircraft seat, occupant, and restraint system has been developed as an aid to the development of crashworthy seats and restraint systems for general aviation aircraft. The occupant model consists of eleven rigid mass segments whose dimensions and inertial properties have been determined from studies of human body anthropometry and kinematics. The seat model is made up of beam and membrane elements with provision for simulating plastic behavior by the introduction of plastic hinges in the beams.

A user-oriented computer program called Seat Occupant Model-Light Aircraft (SOM-LA) based on the three-dimensional model has been developed for use by engineers concerned with design and analysis of general aviation seats and restraint systems in that detailed descriptions of both are used as input. The response of the seat and occupant, restraint system loads, and various injury criteria are predicted for any given set of crash conditions.

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1.0 INTRODUCTION

The design of crashworthy seats and restraint systems for aircraft presents a complex engineering problem, the solution of which can be greatly aided by sufficiently rigorous analytical techniques. The crash environment can vary widely from one accident to another such that a great number of conditions must be evaluated to establish those critical to occupant survival. For example, the restraint system must limit the movement of the occupant sufficiently to eliminate the possibility of head strike on rigid cockpit structure. Also, the relatively low tolerance of the human body to accelerations in a direction parallel to the spine requires the consideration of vertical impact forces which are usually present and often significant in aircraft crashes. It is obvious that a very strong seat is not a valid solution for this problem, since it would not only incur serious weight penalties but would transmit high vertical impact forces directly to the occupant. Rather, the design of a crashworthy seat for aircraft includes the capacity to absorb energy through controlled deformation in the vertical direction, thus lowering the accompanying loads.

In the initial phases it is desirable to evaluate, in some detail, existing seats and restraint systems in their surrounding cockpits, thus establishing existing weaknesses. It is then desirable to make modifications and to evaluate the effect of these modifications on improving the survivability of the system. These evaluations must be conducted for a great many of the possible crash environments, thus constituting a relatively large matrix. It is apparent that testing would be extremely expensive and require a great deal of time, since design modifications would have to be developed and fabricated prior to testing. Therefore, an analytical technique, such as was developed in this program, is required.

A number of one-, two-, and three-dimensional mathematical models of the human body have been developed for crash survivability analysis. These simulation models have generally been

intended for use in evaluation of automobile interior design with respect to injuries caused by secondary impacts, such as the three-dimensional models described in references 1 through 3. Seats have been represented in a very simple manner because in automobiles the role of the seat design in determining occupant survival is minimal. A simulation model intended specifically for this application was, therefore, required.

The development of a three-dimensional mathematical model of a light aircraft seat, occupant, and restraint system is described in this report. This model forms the basis for a simulation computer program that has been written specifically for use in crashworthy design and analysis of light aircraft seats and restraint systems. The program has been organized so as to minimize the volume and complexity of input data and to focus on seat and restraint system design parameters. The effort described herein was performed for the Federal Aviation Administration, Systems Research and Development Service, under Contract DOT-FA72WA-3101, which commenced in August 1972.

2.0 MATHEMATICAL MODEL

The three-dimensional mathematical model includes a lumped-parameter representation of the vehicle occupant and a finite element seat. Interface between the seat and occupant is provided by seat cushions and a restraint system, consisting of a lap belt and, if desired, a single-strap or double-strap shoulder harness. The response of the occupant and seat can be predicted for any given set of aircraft impact conditions, including the initial velocity and attitude and the input acceleration.

This section provides a discussion of the development of the occupant and seat models, including details of the approach to formulating the equations of motion and of the technique used for their solution. The first part of this section discusses the development of the equations, and subsequent parts cover particular aspects of the equations, specifically the body joint model and the treatment of external forces.

2.1 OCCUPANT MODEL

The mathematical model of the aircraft occupant is made up of 11 rigid segments, as shown in figure 1. This number is thought to represent the minimum that will permit accurate, meaningful simulation. A greater number might possibly improve the accuracy of simulation but would, in turn, increase program execution cost. Arm and leg segments are included to enable prediction of injuries to these extremities. Although leg and arm injuries, in themselves, may not be as serious as head or chest injuries, they may prevent escape from a stricken aircraft and the potential hazard of postcrash fire.

Each of the body joints, with the exception of the elbow and knee joints, possesses three rotational degrees of freedom. Because of the hinge-type motion of a forearm or lower leg relative to an upper arm or thigh, respectively, the position of each of these segments is described by one additional angular coordinate. Therefore, the occupant system possesses a total of 28 degrees of freedom.

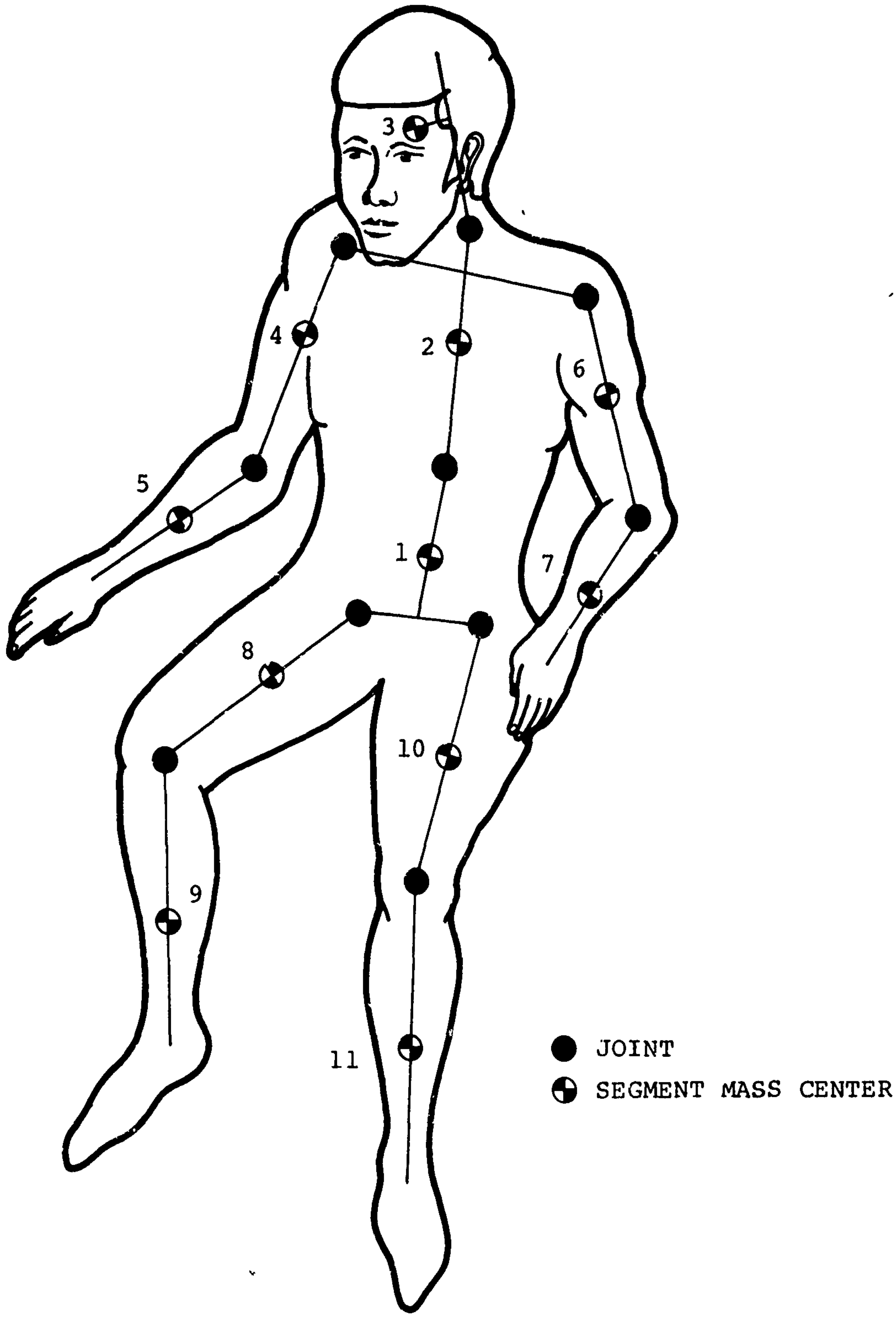


Figure 1. Eleven-Mass Occupant Model.

2.1.1 Equations of Motion

2.1.1.1 Coordinate Systems

Fixed at the center of mass of each of the 11 segments is a right-handed cartesian coordinate system. For segment n (n = 1, 2, ..., 11) the local coordinate system is denoted by axes (x_n, y_n, z_n). Positive directions are defined such that when the body is seated as shown in figure 2, with the torso and head upright, the upper arms parallel to the torso, and the elbows and knees bent at right angles, x_n is positive forward, y_n to the left, and z_n upward.

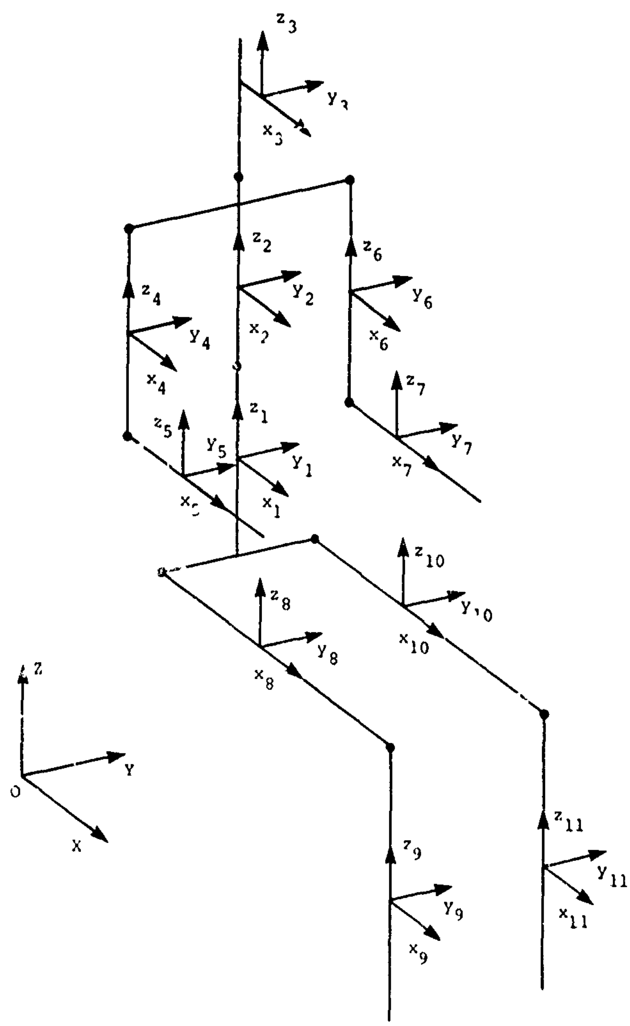


Figure 2. Segment-Fixed Local Coordinate Systems.

In order to describe a general position of the body, it is necessary to relate the orientation of each segment (x_n, y_n, z_n) to the inertial system (X, Y, Z). The angular relationship between the local, segment-fixed coordinate and the inertial system can be expressed by the transformation

$$\begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix} \cdot \begin{Bmatrix} x_n \\ y_n \\ z_n \end{Bmatrix} \tag{1}$$

Because three angular coordinates can be used to define the rotation of a given segment, it is convenient to utilize a set of coordinates that will suffice as generalized coordinates in the formulation of the equations of motion. A system of Eulerian angles provides a convenient set of three independent angular coordinates. Assuming that the local (x_n, y_n, z_n) system is initially coincident with the inertial (X, Y, Z) system, the Euler angles are a series of three rotations, which, when performed in the proper sequence, permit the system to attain any orientation and uniquely define that position. The particular set of Euler angles selected for use here is illustrated in figure 3 and defined, as follows:

1. A positive rotation ψ about the Z-axis, resulting in the primed (x', y', z') system.

2. A positive rotation θ about the y'-axis resulting in the double-primed (x", y", z") system.

3. A positive rotation ϕ about the x"-axis resulting in the final (x, y, z) system.

In order to determine the elements of the transformation matrix $[T^n]$, it is necessary to consider the matrix equations that indicate the three individual rotations described above. Referring again to these definitions of ψ, θ, and ϕ, the following equations are obtained:

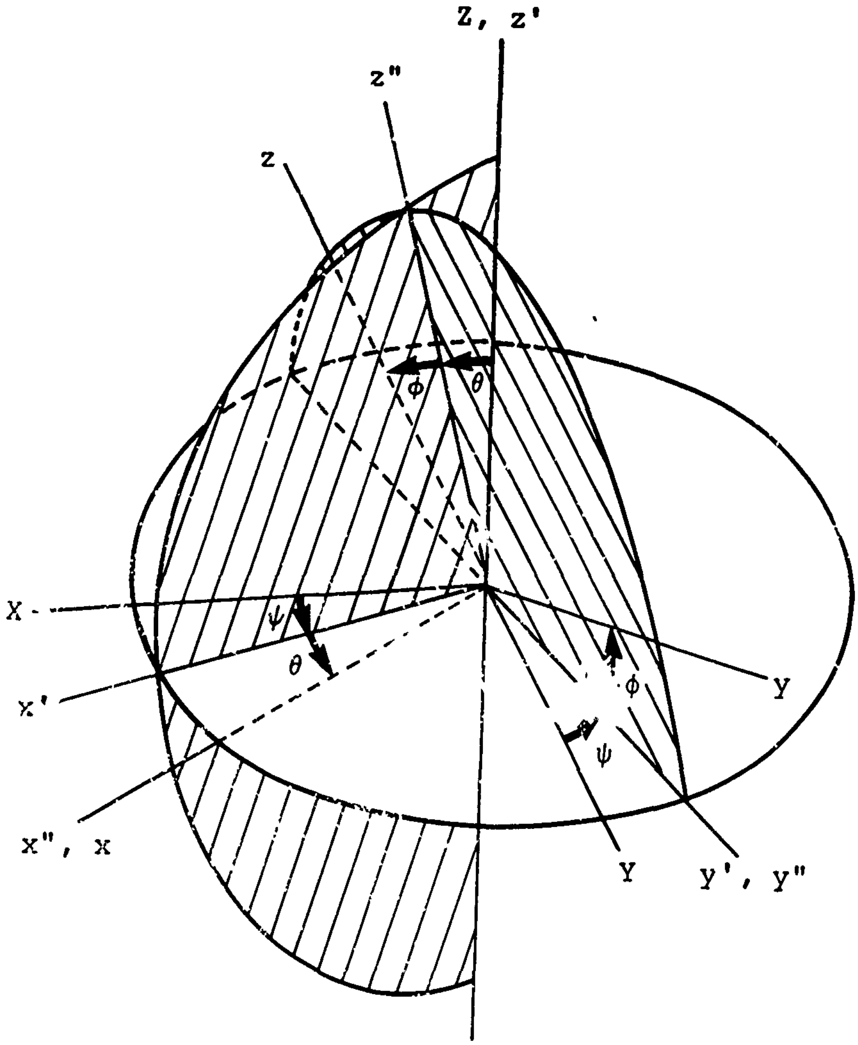


Figure 3. The Euler Angles.

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} \tag{2}$$

$$\begin{Bmatrix} x' \\ y' \\ z' \end{Bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} x'' \\ y'' \\ z'' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{4}$$

Writing equations (2) - (4) in abbreviated form

$$\{R\} = [\psi]\ \{r'\}$$

$$\{r'\} = [\theta]\ \{r''\}$$

$$\{r''\} = [\phi]\ \{r\}$$

or

$$\{R\} = [\psi]\ [\theta]\ [\phi]\ \{r\} \tag{5}$$

where $\{R\}$ represents the components of a vector in the inertial system and $\{r\}$ represents the same vector in the final (x, y, z) system. Performing the matrix multiplications indicated in equation (5), the elements of the transformation matrix in equation (1) are obtained:

$$T_{11}^{\ n} = \cos\psi_n \cos\theta_n$$

$$T_{12}^{\ n} = \cos\psi_n \sin\theta_n \sin\phi_n - \sin\psi_n \cos\phi_n$$

$$T_{13}^{\ n} = \cos\psi_n \sin\theta_n \cos\phi_n + \sin\psi_n \sin\phi_n$$

$$T_{21}^{\ n} = \sin\psi_n \cos\theta_n$$

$$T_{22}^{\ n} = \sin\psi_n \sin\theta_n \sin\phi_n + \cos\psi_n \cos\phi_n$$

$$T_{23}^{\ n} = \sin\psi_n \sin\theta_n \cos\phi_n - \cos\psi_n \sin\phi_n$$

$$T_{31}^{\ n} = -\sin\theta_n$$

$$T_{32}^{\ n} = \cos\theta_n \sin\phi_n$$

$$T_{33}{}^n = \cos\theta_n \cos\phi_n$$

$$\text{for } n = 1, 2, 3, 4, 6, 8, 10 \tag{6}$$

The additional constraint of hinge-type rotation, at the elbows and knees, requires the use of one additional angular coordinate to define the position of each of the forearm and lower leg segments. Referring to figure 4, the angular position of the forearm segments (ℓ = 5, 7) is given by

$$\begin{Bmatrix} X_\ell \\ Y_\ell \\ Z_\ell \end{Bmatrix} = \begin{bmatrix} \\ T^n \\ \\ \end{bmatrix} \begin{bmatrix} \sin\alpha_\ell & 0 & \cos\alpha_\ell \\ 0 & 1 & 0 \\ -\cos\alpha_\ell & 0 & \sin\alpha_\ell \end{bmatrix} \begin{Bmatrix} x_\ell \\ y_\ell \\ z_\ell \end{Bmatrix} \tag{7}$$

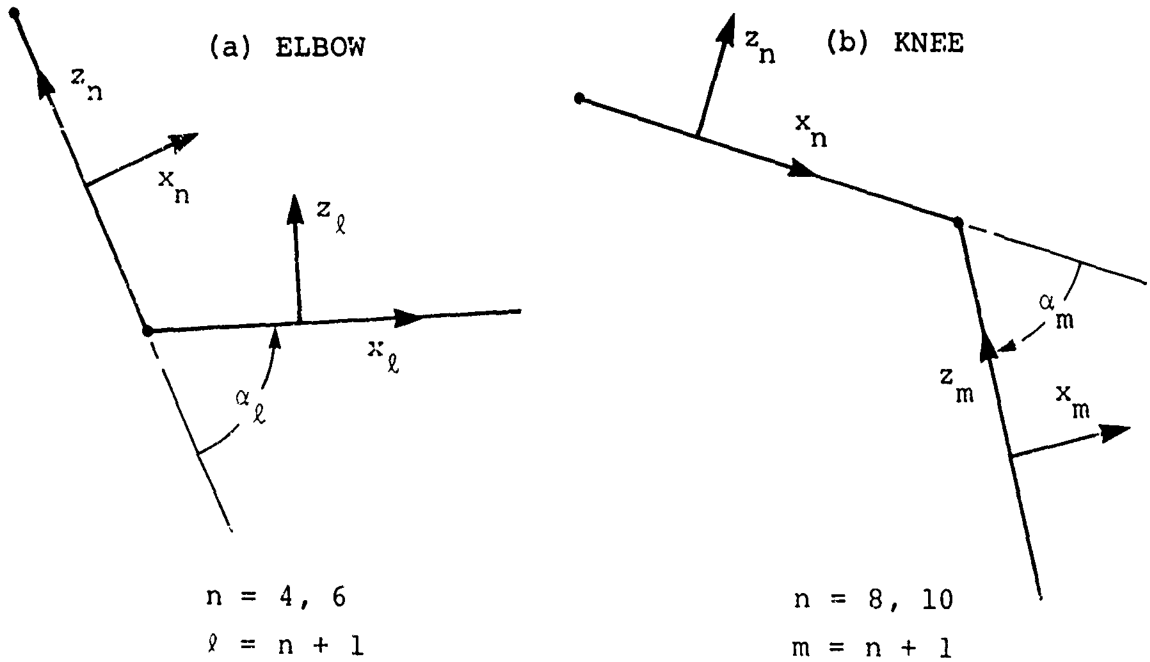


Figure 4. Definition of Angular Coordinates α for Elbows and Knees.

and the lower leg segments (m = 9, 11) by

$$\begin{Bmatrix} X_m \\ Y_m \\ Z_m \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix} \begin{bmatrix} \sin \alpha_m & 0 & -\cos \alpha_m \\ 0 & 1 & 0 \\ \cos \alpha_m & 0 & \sin \alpha_m \end{bmatrix} \begin{Bmatrix} x_m \\ y_m \\ z_m \end{Bmatrix} \tag{8}$$

From equations (6) and (7) the elements of the transformation matrix for the forearms (sements 5 and 7) are written as:

$$T_{11}^{\ell} = \cos \psi_n \cos \theta_n \sin \alpha_\ell - \cos \psi_n \sin \theta_n \cos \phi_n \cos \alpha_\ell - \sin \psi_n \sin \phi_n \cos \alpha_\ell$$

$$T_{12}^{\ell} = \cos \psi_n \sin \theta_n \sin \phi_n - \sin \psi_n \cos \phi_n$$

$$T_{13}^{\ell} = \cos \psi_n \cos \theta_n \cos \alpha_\ell + \cos \psi_n \sin \theta_n \sin \phi_n \sin \alpha_\ell + \sin \psi_n \sin \phi_n \sin \alpha_\ell$$

$$T_{21}^{\ell} = \sin \psi_n \cos \theta_n \sin \alpha_\ell - \sin \psi_n \sin \theta_n \cos \phi_n \cos \alpha_\ell - \cos \psi_n \sin \phi_n \cos \alpha_\ell$$

$$T_{22}^{\ell} = \sin \psi_n \sin \theta_n \sin \phi_n + \cos \psi_n \cos \phi_n$$

$$T_{23}^{\ell} = \sin \psi_n \cos \theta_n \cos \alpha_\ell + \sin \psi_n \sin \theta_n \cos \phi_n \sin \alpha_\ell - \cos \psi_n \sin \phi_n \sin \alpha_\ell$$

$$T_{31}^{\ell} = - \sin \theta_n \sin \alpha_\ell - \cos \theta_n \cos \phi_n \cos \alpha_\ell$$

$$T_{32}^{\ell} = \cos \theta_n \sin \phi_n$$

$$T_{33}^{\ell} = - \sin \theta_n \cos \alpha_\ell + \cos \theta_n \cos \phi_n \sin \alpha_\ell \tag{9}$$

From equations (6) and (8) the elements of the transformation matrix for the legs (segments 9 and 11) are obtained:

$$T_{11}^{m} = \cos\psi_n \cos\theta_n \sin\alpha_m + \cos\psi_n \sin\theta_n \cos\phi_n \cos\alpha_m + \sin\psi_n \sin\phi_n \cos\alpha_m$$

$$T_{12}^{m} = \cos\psi_n \sin\theta_n \sin\phi_n - \sin\psi_n \cos\phi_n$$

$$T_{13}^{m} = \cos\psi_n \cos\theta_n \cos\alpha_m + \cos\psi_n \sin\theta_n \cos\phi_n \sin\alpha_m + \sin\psi_n \sin\phi_n \sin\alpha_m$$

$$T_{21}^{m} = \sin\psi_n \cos\theta_n \sin\alpha_m + \sin\psi_n \sin\theta_n \cos\phi_n \cos\alpha_m - \cos\psi_n \sin\phi_n \cos\alpha_m$$

$$T_{22}^{m} = \sin\psi_n \sin\theta_n \sin\phi_n + \cos\psi_n \cos\phi_n$$

$$T_{23}^{m} = -\sin\psi_n \cos\theta_n \cos\alpha_m + \sin\psi_n \sin\theta_n \cos\phi_n \sin\alpha_m - \cos\psi_n \sin\phi_n \sin\alpha_m$$

$$T_{31}^{m} = -\sin\theta_n \sin\alpha_m + \cos\theta_n \cos\phi_n \cos\alpha_m$$

$$T_{32}^{m} = \cos\theta_n \sin\phi_n$$

$$T_{33}^{m} = \sin\theta_n \cos\alpha_m + \cos\theta_n \cos\phi_n \sin\alpha_m \qquad (10)$$

Having developed the relationships expressed in equations (1) through (10), the position of the occupant can be described by the following set of generalized coordinates:

$$
\begin{array}{lll}
q_1 = X_1 & q_{11} = \theta_3 & q_{21} = \psi_8 \\
q_2 = Y_1 & q_{12} = \phi_3 & q_{22} = \theta_8 \\
q_3 = Z_1 & q_{13} = \psi_4 & q_{23} = \phi_8 \\
q_4 = \psi_1 & q_{14} = \theta_4 & q_{24} = \alpha_9 \\
q_5 = \theta_1 & q_{15} = \phi_4 & q_{25} = \psi_{10} \\
q_6 = \phi_1 & q_{16} = \alpha_5 & q_{26} = \theta_{10} \\
q_7 = \psi_2 & q_{17} = \psi_6 & q_{27} = \phi_{10} \\
q_8 = \theta_2 & q_{18} = \theta_6 & q_{28} = \alpha_{11} \\
q_9 = \phi_2 & q_{19} = \phi_6 & \\
q_{10} = \psi_3 & q_{20} = \alpha_7 &
\end{array}
\qquad (11)
$$

The above coordinates include the cartesian coordinates of the mass center of segment 1 (X_1, Y_1, Z_1), selected as a reference point on the body, seven sets of Eulerian angles, and the four additional angular coordinates for the elbows and knees.

2.1.1.2 Lagrange's Equations

The response of the occupant system is described by Lagrange's equations of motion, which are written for the 28 generalized coordinates. The equations are developed according to

$$
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \qquad (j = 1, 2, \ldots, 28) \qquad (12)
$$

where L is the Lagrangian function

$$L = T - V \tag{13}$$

t represents time, Q_j are the generalized forces not derivable from a potential function. (Forces that are derivable from a potential function are obtained from L and T and V are the system kinetic and potential energies, respectively.)

Because the system being treated does not involve any velocity-dependent potentials, equation (12) can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j \qquad (j = 1, 2, \ldots, 28) \tag{14}$$

The system kinetic energy contains both translational and rotational parts:

$$T = \frac{1}{2} \sum_{n=1}^{11} M_n \left[(\dot{X}_n)^2 + (\dot{Y}_n)^2 + (\dot{Z}_n)^2\right] + \frac{1}{2} \sum_{n=1}^{11} \left(I_{x_n} \omega_{x_n}{}^2 + I_{y_n} \omega_{y_n}{}^2 + I_{x_n} \omega_{z_n}{}^2\right) \tag{15}$$

where M_n is the mass of segment n and I_{x_n}, I_{y_n}, and I_{z_n} are mass moments of inertia of segment n with respect to the local coordinate aces (x_n, y_n, z_n), assumed to be principal moments of inertia.

The absolute velocities of the 11 mass segments required for the translational kinetic energy must, of course, be written as functions of the generalized coordinates and generalized velocities in order to use equation (14). The derivation of these velocities is presented in appendix A. The angular velocity components (ω_{x_n}, ω_{y_n}, ω_{z_n}) seen in equation (15) are parallel to the local (x_n, y_n, z_n) coordinate systems. These angular velocity components cannot be used directly in Lagrange's equations because they do not correspond to the time derivatives of any

set of coordinates that specify the position of the segment. They must be written as functions of the generalized coordinates, using the generalized angular velocites ($\dot{\underline{\psi}}_n$, $\dot{\underline{\theta}}_n$, $\dot{\underline{\phi}}_n$), which are parallel to the axes Z, y_n', and x_n'', respectively.

An arbitrary angular velocity of segment n, $\underline{\omega}_n$, can be expressed as a function of the generalized angular velocities according to

$$\underline{\omega}_n = \dot{\underline{\psi}}_n + \dot{\underline{\theta}}_n + \dot{\underline{\phi}}_n \tag{16}$$

Referring to figure 3, $\dot{\underline{\psi}}_n$, $\dot{\underline{\theta}}_n$, and $\dot{\underline{\phi}}_n$ do not, in general, form a mutually perpendicular vector triad. ($\dot{\underline{\psi}}$ and $\dot{\underline{\phi}}$ are both perpendicular to $\dot{\underline{\theta}}$ but are not necessarily perpendicular to each other.) However, they can be considered as a nonorthogonal set of components of $\underline{\omega}$ since their vector sum is equal to $\underline{\omega}$. Summing the orthogonal projections of $\dot{\underline{\psi}}_n$, $\dot{\underline{\theta}}_n$, and $\dot{\underline{\phi}}_n$ on the (x_n, y_n, y_n) axes yields the angular velocity components required for the kinetic energy expression:

$$\omega_{x_n} = \dot{\phi}_n - \dot{\psi}_n \sin \theta_n$$

$$\omega_{y_n} = \dot{\psi}_n \cos \theta_n \sin \phi_n + \dot{\theta}_n \cos \phi_n$$

$$\omega_{z_n} = \dot{\psi}_n \cos \theta_n \cos \phi_n - \dot{\theta}_n \sin \phi_n \tag{17}$$

All quantities required to develop the system kinetic energy are now available, and the entire expression is written in appendix B.

The system potential energy is simply gravitational potential, which is written as

$$V = \sum_{n=1}^{11} M_n g \, (Z_n - Z_{n_o}) \tag{18}$$

where g is the acceleration due to gravity and Z_{n_o} is an arbitrary datum. The potential energy expression is expanded in appendix C.

2.1.1.3 Matrix Equations

For purposes of computation, the equations of motion are rewritten in the following form:

$$[A(q)]\ \{\ddot{q}\} = \{B(\dot{q},q)\} + \{P(q)\} + \{R(\dot{q},q)\} + \{Q(\dot{q},q)\} \qquad (19)$$

where the elements of the inertia matrix [A] and the vector {B} are derived from the kinetic energy derivatives of Lagrange's equations. In other words,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \sum_{k=1}^{28} A_{jk}\ddot{q}_k$$

$$- B_j\ (\dot{q}_1,\ \ldots,\ \dot{q}_{28},\ q_1,\ \ldots,\ q_{28})$$

$$(j = 1,\ 2,\ \ldots,\ 28) \qquad (20)$$

The elements of [A] and {B} are given in appendix D. The force vector {P} is derived from the system potential energy according to

$$P_j\quad (q_1,\ \ldots,\ q_{28}) = -\frac{\partial V}{\partial q_j}\quad (j = 1,\ 2,\ \ldots,\ 28) \qquad (21)$$

The elements of {P} are presented in appendix E. Both {R} and {Q} are vectors of generalized forces derived from the right-hand side of Lagrange's equations. The vector {R} describes the resistance of the body joints to rotation and is discussed in detail in section 2.1.2. {Q} is the vector of generalized external forces and is discussed in detail in section 2.1.3.

2.1.2 Joint Resistance

The form of the joint resistance vector {R} in equation (19) depends on the user's selection of occupant - either dummy or human. Although both joint models contain the same types of elements, a nonlinear torsional spring and a viscous torsional damper, the function of each of these elements determines the type of occupant.

The 10 body joints, illustrated in figure 5, are defined as follows:

Joint 1 - Back, between 12th thoracic and 1st lumbar vertebrae

Joint 2 - Neck, between 7th cervical and 1st thoracic vertebrae

Joint 3 - Right shoulder

Joint 4 - Right elbow

Joint 5 - Left shoulder

Joint 6 - Left elbow

Joint 7 - Right hip

Joint 8 - Right knee

Joint 9 - Left hip

Joint 10 - Left knee

The angular displacement of joint i from its reference position (figure 2) is given by β_i. If (i_m, j_m, k_m) and (i_n, j_n, k_n) are triads of unit vectors in the local coordinate systems of two adjacent segments connected at joint i, as shown in figure 6, the joint angle is given by

$$\beta_i = \cos^{-1} (k_m \cdot k_n) \qquad (22)$$

where $(k_m \cdot k_n)$ is the scalar product. Considering the geometry of the occupant model in the reference position, the β_i for the 10 joints are given by

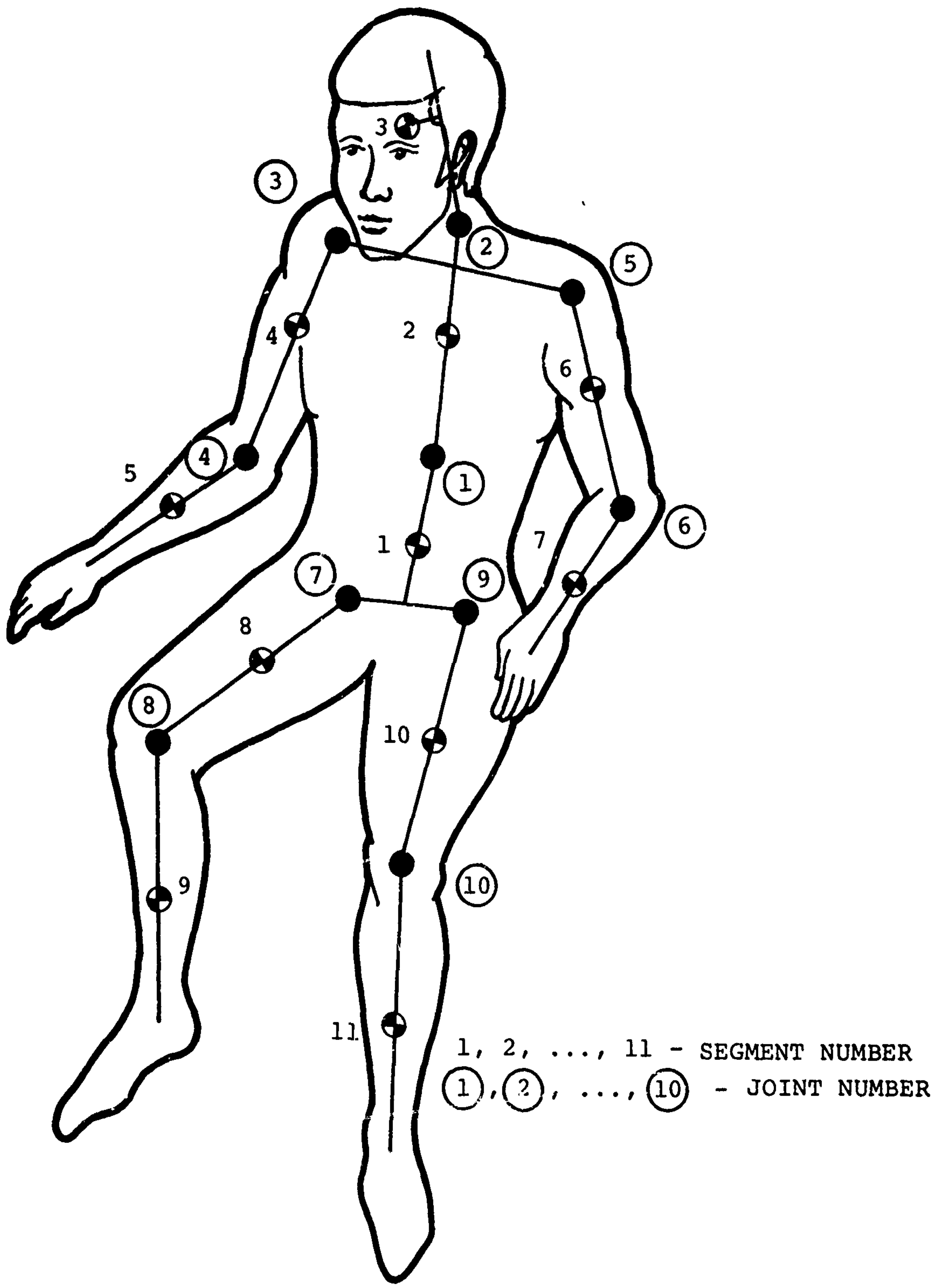


Figure 5. Joint Numbering System.

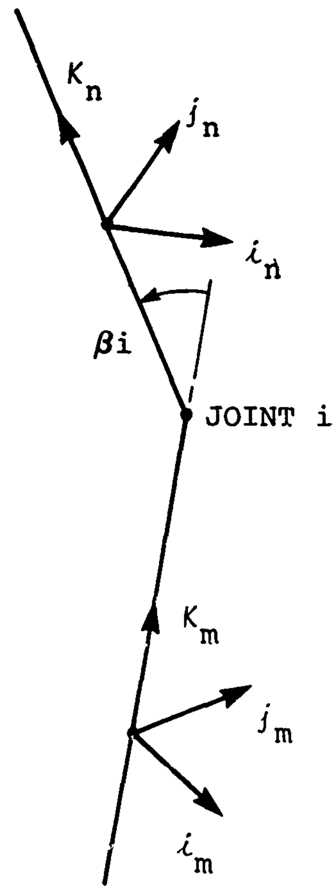


Figure 6. Joint Angle β_i Between Segments m and n.

$$\beta_1 = \cos^{-1} (k_1 \cdot k_2)$$

$$\beta_2 = \cos^{-1} (k_2 \cdot k_3)$$

$$\beta_3 = \cos^{-1} (k_2 \cdot k_4)$$

$$\beta_4 = \alpha_5$$

$$\beta_5 = \cos^{-1} (k_2 \cdot k_6)$$

$$\beta_6 = \alpha_7$$

$$\beta_7 = \cos^{-1}(\imath_1 \cdot \imath_8)$$

$$\beta_8 = \alpha_9$$

$$\beta_9 = \cos^{-1}(\imath_1 \cdot \imath_{10})$$

$$\beta_{10} = \alpha_{11} \tag{23}$$

If at each joint i, a moment M_i and a torsional damper with coefficient J_i act to resist motion of the joint, then the virtual work done on the system as each joint i undergoes a virtual displacement $\delta\beta_i$ is

$$\delta W = - \sum_{i=1}^{10} (M_i \, \delta\beta_i + J_i \, \dot{\beta}_i \, \delta\beta_i) \tag{24}$$

Since the β_i are functions of the generalized coordinates q_j, the virtual displacements $\delta\beta_i$ can be expressed in terms of corresponding virtual displacements of the q_j. In general, such an expression would take the form

$$\delta\beta_i = \sum_{j=1}^{28} \frac{\partial\beta_i}{\partial q_j} \delta q_j \qquad (i=1, 2, \ldots, 10) \tag{25}$$

where the partial derivatives $\partial\beta_i/\partial q_j$ are functions of the generalized coordinates. Substituting into equation (24) gives

$$\delta W = - \sum_{i=1}^{10} \sum_{j=1}^{28} (M_i + J_i \, \dot{\beta}_i) \frac{\partial\beta_i}{\partial q_j} \delta q_j \tag{26}$$

Changing the order of summation, equation (26) can be written in the general form

$$\delta W = \sum_{j=1}^{28} Q_j \, \delta q_j \tag{27}$$

where Q_j are the generalized forces acting on the system. As seen in equation (19), the generalized forces are being treated as two distinct types: joint resistance forces and external forces. Since the joint resistance terms are being treated here, the generalized joint forces referred to as R_j will be considered alone. Equation (27) becomes, more specifically

$$\delta W = \sum_{j=1}^{28} R_j \, \delta q_j \tag{28}$$

and, from equation (26), R_j can be written

$$R_j = - \sum_{i=1}^{10} (M_i + J_i \, \dot{\beta}_i) \frac{\partial \beta_i}{\partial q_j} \qquad (j=1, 2, \ldots, 28) \tag{29}$$

The elements of vector {R} and their derivation, using equation (29), are presented in appendix F.

As mentioned earlier in this section, the type of occupant is determined by the relative contributions of M_i and J_i to the R_j terms. For the dummy joint the resisting torque M_i is constant throughout the normal range of joint motion and increases rapidly along a third-order curve to a higher value at the limiting displacement β_{S_i}, as shown in figure 7. The normal values M_{D_i} are set equal to those resulting from the joint-tightening procedure of SAE Recommended Practice, Anthropomorphic Test Device for Dynamic Testing - SAE J963. That is, the body joints will just support a 1G load in the reference (seated) position, with the exception of the torso joints, which will support a 2G load. In addition to M_i, a small viscous damping term with constant J_i is included for energy dissipation.

The resistance of the human joint consists of up to three terms. The primary resisting force during normal joint rotation is a viscous damping term with constant coefficient J_i. In a manner similar to the case of the dummy, a resisting torque is applied at the limit of the joint range of motion, as shown in

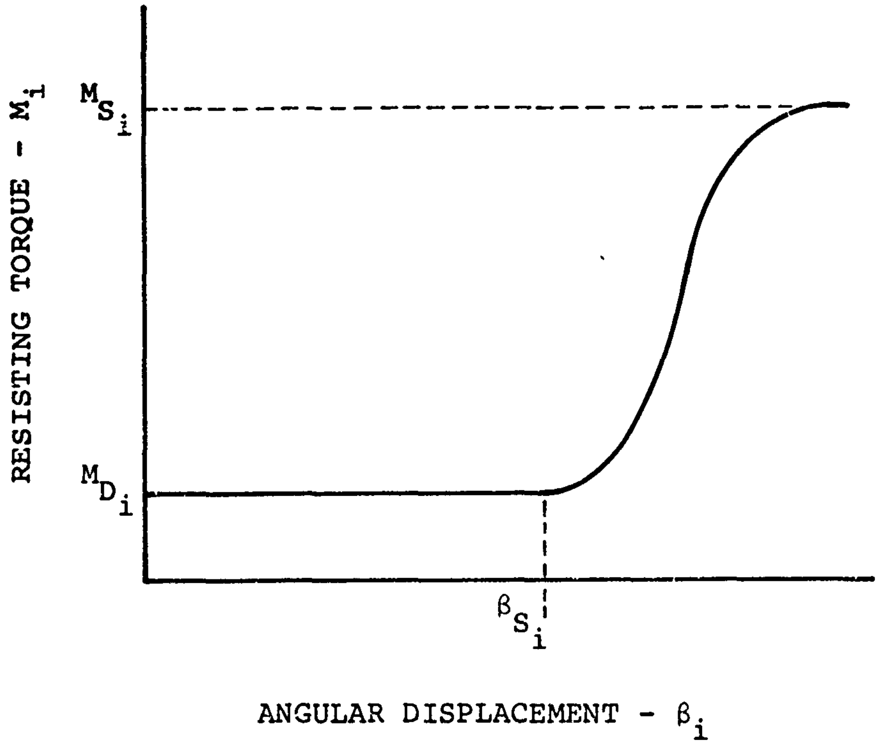


Figure 7. Dummy Joint Resisting Torque.

figure 8(a). An additional term used to simulate muscle tone is the moment M', which drops to zero after a small angular displacement from the initial position, provided that the crash deceleration is sufficient to overcome it (figure 8(b)). A completely relaxed occupant, usually the worst case with regard to injury, can be simulated by setting this last joint resistance term to zero.

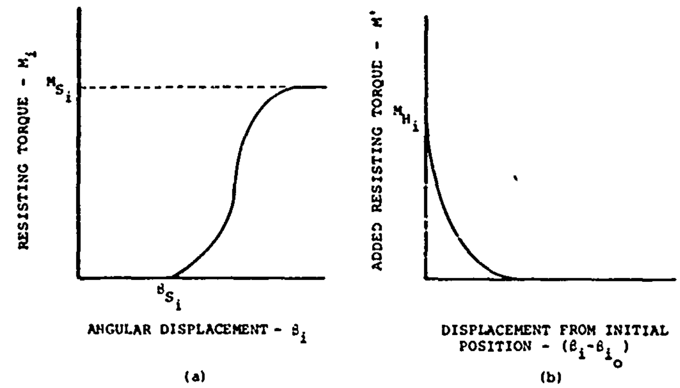


Figure 8. Human Joint Resisting Torques: (a) Displacement-Limiting Moment; (b) Muscular Resistance.

2.1.3 External Forces

The vector of generalized external forces {Q} is developed in a manner similar to that discussed in the previous section for the joint resistance vector. The resultant external force $\underline{F}_i$ acting on segment i is given by

$$\underline{F}_i = F_{X_i}\, i + F_{Y_i}\, j + F_{Z_i}\, k \tag{30}$$

where F_{X_i}, F_{Y_i}, and F_{Z_i} are components in the inertial (X, Y, Z) system. Let the absolute position of the point P_i on segment i, where the resultant force acts, be represented by

$$\underline{r}_{P_i} = X_{P_i}\, i + Y_{P_i}\, j + Z_{P_i}\, k \tag{31}$$

As the resultant force applied to each segment i undergoes a virtual displacement δr_{P_i}, having components (δX_{P_i}, δY_{P_i}, δZ_{P_i}), the virtual work on the system done by the $\underline{F}_i$ is

$$\delta W = \sum_{i=1}^{11} (F_{X_i} \delta X_{P_i} + F_{Y_i} \delta Y_{P_i} + F_{Z_i} \delta Z_{P_i}) \tag{32}$$

Writing the virtual displacement components in terms of the generalized coordinates q_j:

$$\delta X_{P_i} = \sum_{j=1}^{28} \frac{\partial X_{P_i}}{\partial q_j} \delta q_j$$

$$\delta Y_{P_i} = \sum_{j=1}^{28} \frac{\partial Y_{P_i}}{\partial q_j} \delta q_j$$

$$\delta Z_{P_i} = \sum_{j=1}^{28} \frac{\partial Z_{P_i}}{\partial q_j} \delta q_j \tag{33}$$

results in

$$\delta W = \sum_{j=1}^{28} \sum_{i=1}^{11} (F_{X_i} \frac{\partial X_{P_i}}{\partial q_j} + F_{Y_i} \frac{\partial Y_{P_i}}{\partial q_j} + F_{Z_i} \frac{\partial Z_{P_i}}{\partial q_j}) \delta q_j \tag{34}$$

Using equation (27)

$$\delta W = \sum_{j=1}^{28} Q_j \delta q_j$$

yields the components of the generalized external force vector:

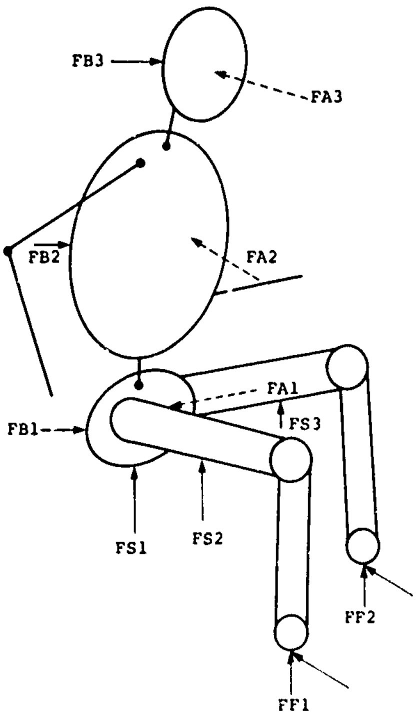
$$Q_j = \sum_{i=1}^{11} (F_{X_i} \frac{\partial X_{P_i}}{\partial q_j} + F_{Y_i} \frac{\partial Y_{P_i}}{\partial q_j} + F_{Z_i} \frac{\partial Z_{P_i}}{\partial q_j}) \tag{35}$$

The components of {Q} derived from equation (35) are presented in appendix G.

The external forces acting on the 11 body segments can be characterized as either contact forces or restraint forces. These forces are discussed in further detail in the sections following.

2.1.3.1 Contact Forces

The contact forces exerted on the occupant include all those, except the restraint forces. As illustrated in figure 9, these forces are exerted by the cushions, floor, and an optional air bag restraint. These forces are all assumed to pass through segment mass centers, with the exception of the floor forces and the force of the seat cushion on segment 1. The floor forces act on the ends of leg segments, and the force of the seat cushion on segment 1 passes through a point midway between the hip joints. This assumption effects a considerable simplification in the model, for if the position of the point of force application, $\underline{r}_{p_i}$, is the position of a segment mass center, it is available directly from the solution of the equations of motion.



FORCE IDENTIFICATION	
FB1, FB2:	BACK CUSHION
FB3:	HEAD REST
FS1, FS2, FS3:	SEAT CUSHION
FF1, FF2:	FLOOR
FA1, FA2, FA3:	AIR BAG (OPTIONAL)

Figure 9. External Forces of Cushions, Floor, and Air Bag.

All contact forces are calculated by first determining the penetration of a contact surface on the occupant into a surface with known force-deflection characteristics. Using the seat cushion force as an example, the pertinent dimensions of the seat and the parameters required to determine the penetration of the abdomino-pelvic segment (segment 1) into the cushion are illustrated in figure 10. X_P and Z_P are coordinates of the center of the contact surface of segment 1, and R_1 is the radius of the contact surface in the $(x_1 - z_1)$ plane. (Although this contact surface is an ellipsoid, cross-sections parallel to the $(x_1 - z_1)$ plane are circular. The dimensions of the contact surfaces will be discussed in a later section.) The position of the seat pan is defined by its height Z_s above the origin of the aircraft coordinate system and the angle θ_s that it makes with the aircraft $(X_A - Y_A)$ plane. The unloaded thickness of the seat cushion is t_e, and the loaded thickness beneath segment 1 is t. Summing the dimensions in the Z_A direction gives

$$Z_P = Z_s + (R_1 + t)/\cos\theta_x + X_P \tan\theta_s \qquad (36)$$

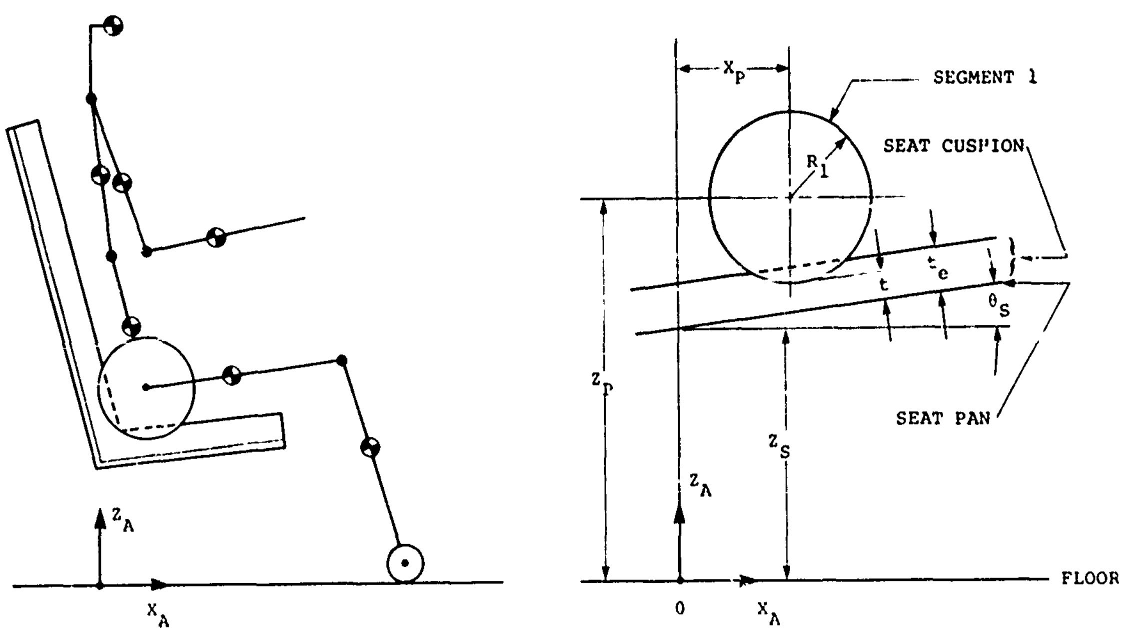


Figure 10. Seat Cushion Deflection.

Solving equation (36) for the cushion thickness,

$$t = (Z_P - Z_S) \cos \theta_S - R_1 - X_P \sin \theta_S \qquad (37)$$

The deflection of the seat cushion is then

$$\delta_c = t_e - t \qquad (38)$$

and the force, which is assumed to act normal to the plane of the seat pan and pass through the center of curvature of the contact surface, is calculated from an input curve of force-versus-deflection. The other contact forces are calculated in the same manner. A small difference exists in the calculation of the air bag force because the air bag surface is cylindrical, but the technique is basically the same.

2.1.3.2 Restraint System Forces

The method used in calculating the forces exerted on the body by the restraint system differs considerably from that described in the preceding section for the contact forces. The primary reason for this difference is that the restraint forces do not act at any fixed points on the occupant, but, rather, the points of application vary with the restraint system geometry.

Although other configurations can be selected by the user, the basic restraint system consists of a lap belt and diagonal shoulder strap. The restraint loads are transmitted to the occupant model through ellipsoidal surfaces fixed to the upper and lower torso segments. These surfaces are shown in figure 11. The locations of the anchor points A_1, A_2, and A_3 and the buckle connection B are determined by user input along with the webbing properties.

The ellipsoidal surfaces are described by

$$x_1^2/a_1^2 + y_1^2/b_1^2 + z_1^2/c_1^2 = 1 \qquad (39)$$

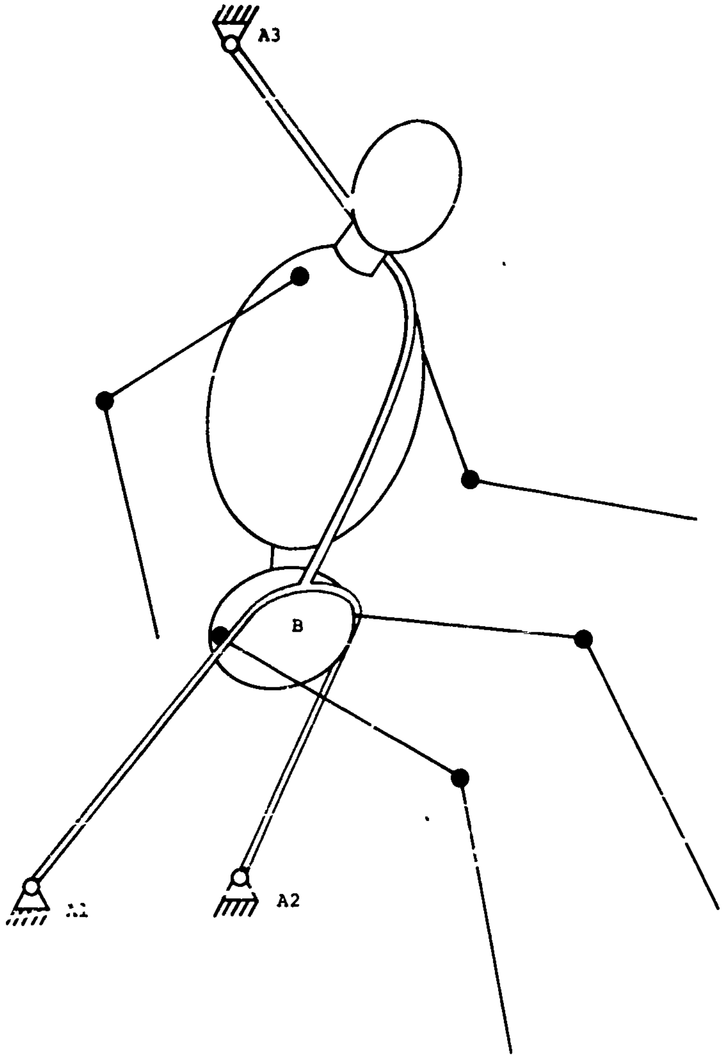


Figure 11. Basic Restraint System Configuration.

for the lower torso, where

$$a_1 = R_1$$

$$b_1 = L_H + R_{16}$$

$$c_1 = R_1$$

and

$$x_2^2/a_2^2 + y_2^2/b_2^2 + z_2^2/c_2^2 = 1 \tag{40}$$

for the upper torso, where

$$a_2 = R_2$$

$$b_2 = L_2/2$$

$$c_2 = L_2/2$$

and these body dimensions are defined in section 2.1.4.

The restraint forces are determined in the same manner for both the upper and lower torso. First, the belt loads are calculated from the displacements of the torso segments, and the resultant force on each segment is then applied at the point along the arc of contact between the belt and the ellipsoidal surface where the force is normal to the surface.

Explaining this procedure in further detail for the restraint system configuration shown in figure 11, for any position of the occupant, the coordinates of the left shoulder, the hips, and the buckle connection B are calculated in the aircraft reference frame. The length of each of the three belt segments (right lap belt, left lap belt, and shoulder strap) is equal to the sum of the free length, or the distance between the appropriate anchor point and a reference point on the hip or shoulder, and the distance along the arc from that hip or shoulder to the buckle. If that length should exceed the equilibrium (zero load) length calculated initially then there is some tensile force in the belt. The resultant force on each segment is the vector sum of the belt forces. Friction between the shoulder belt and chest along the length of the belt is taken into account by reducing the load in the belt between the chest and buckle by a constant

fraction of the load in the free length between the belt and the body surface. The resultant force on the lower or upper torso segment may be written generally as

$$\underline{F} = F_x \, i + F_y \, j + F_z \, k \tag{41}$$

where F_x, F_y, and F_z are components in the local, segment-fixed coordinate system.

To find the point on the segment where $\underline{F}$ is normal to the surface, consider first the equation of an ellipsoid:

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \tag{42}$$

which may also be expressed in functional form as

$$f\ (x, y, z) = x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 \tag{43}$$

where the ellipsoid can be regarded as the level surface f=0 of the function. At any point (x, y, z) on the surface, the gradient of f is normal to the surface. The gradient is given by

$$\text{grad } f = (2x/a^2)\ i + (2y/b^2)\ j + (2z/c^2)\ k \tag{44}$$

and at the point of application of the resultant force, grad f is collinear with $\underline{F}$. Making use of the proportionality between the components of the two vectors,

$$\begin{aligned} F_x &= Cx/a^2 \\ F_y &= Cy/b^2 \\ F_z &= Cz/c^2 \end{aligned} \tag{45}$$

where C is an arbitrary constant. Solving equation (45) for the coordinates (x, y, z) and substituting into equation (42)

$$\left(\frac{F_x a^2}{C}\right)^2 \frac{1}{a^2} + \left(\frac{F_x b^2}{C}\right)^2 \frac{1}{b^2} + \left(\frac{F_z c^2}{C}\right)^2 \frac{1}{c^2} = 1 \tag{46}$$

$$\left(\frac{F_x a}{C}\right)^2 + \left(\frac{F_y b}{C}\right)^2 + \left(\frac{F_z c}{C}\right)^2 = 1 \tag{47}$$

which leads to

$$C^2 = F_x{}^2 a^2 + F_y{}^2 b^2 + F_z{}^2 c^2$$

$$C = \pm\sqrt{F_x{}^2 a^2 + F_y{}^2 b^2 + F_z{}^2 c^2} \tag{48}$$

the point of application of $\underline{F}$ is then

$$x = F_x a^2/C$$

$$y = F_y b^2/C$$

$$z = F_z c^2/C$$

with

$$C = -\sqrt{F_x{}^2 a^2 + F_y{}^2 b^2 + F_z{}^2 c^2} \tag{49}$$

The negative sign on C can be explained by the fact that each coordinate in the local system is opposite in sign to the corresponding component of the resultant force, or

$$x > 0 \text{ if } F_x < 0$$

$$y > 0 \text{ if } F_y < 0$$

$$z > 0 \text{ if } F_z < 0 \tag{50}$$

The capability of the belts to slide relative to the torso surfaces allows simulation of "submarining" under the lap belt, as illustrated in the sequence of figure 12.

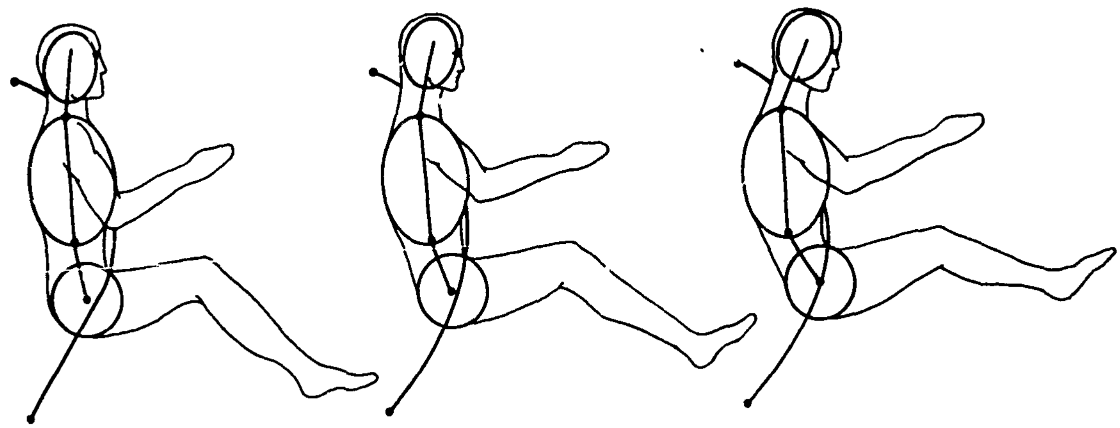


Figure 12. Occupant Submarining.

2.1.4 Occupant Physical Properties

2.1.4.1 Body Segment Dimensions

The basic dimensions of the occupant segments that are required in writing the equations of motion are illustrated in figure 13. The lengths of the segments are, in most cases, effective "link lengths" between joint centers, rather than standard anthropometric dimensions based on external measurements. These lengths are calculated as fixed fractions of total body stature, obtained from references 4 and 5. These fractions, along with the actual segment lengths for a 50th percentile male (69.1-inch stature), are presented in table 1.

The distance of the mass center of segment n from the end nearest the body reference point (C_1) is ρ_n. The distance between the mass center and the far end is given by

$$\bar{\rho}_n = L_n - \rho_n \tag{51}$$

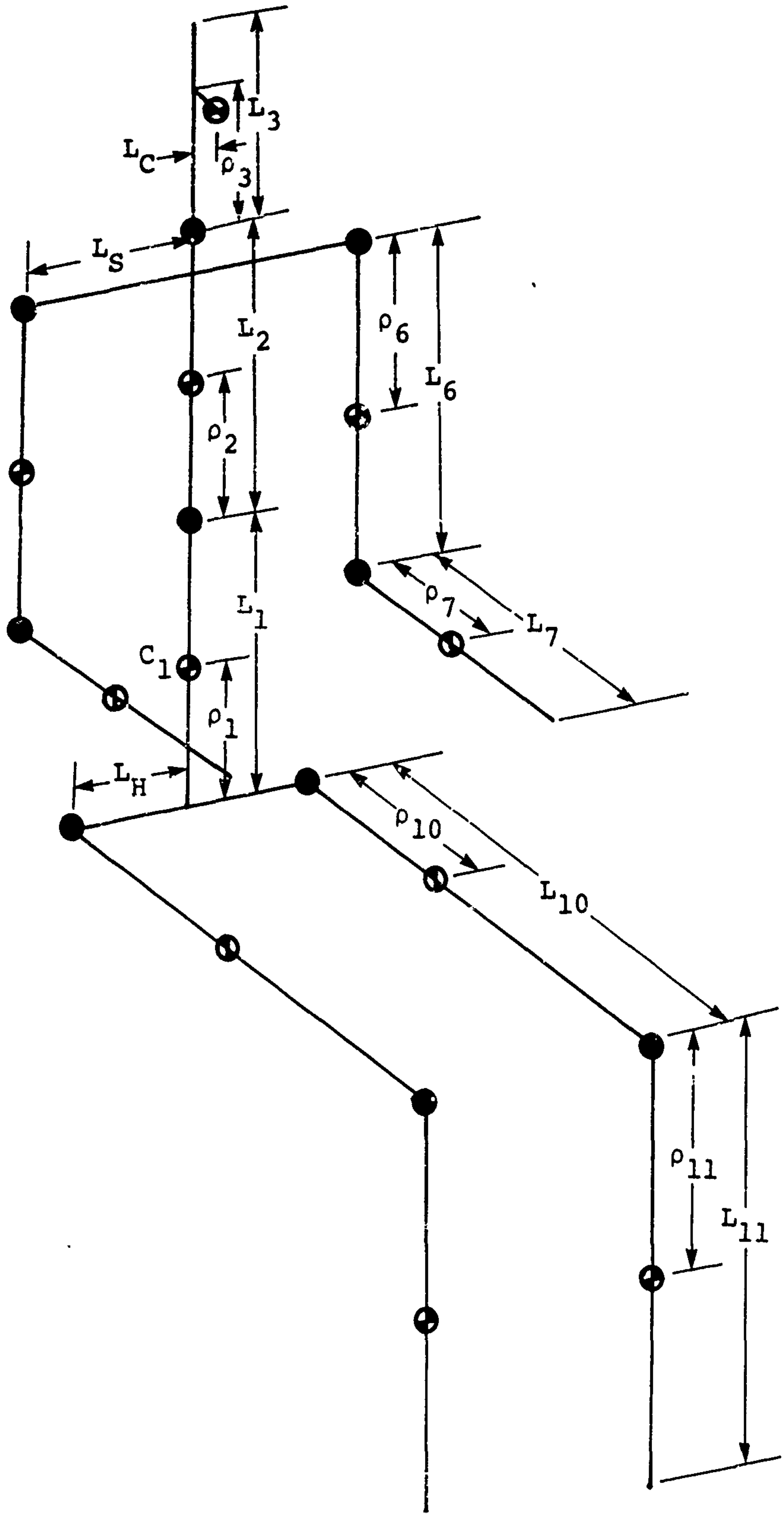


Figure 13. Body Segments Lengths and Mass Center Locations.

TABLE 1. BODY SEGMENT LENGTHS			
Segment	Symbol	Fraction of Stature	Length for 50th Percentile Male (in.)
Lower Torso	L_1	0.1714	11.84
Upper Torso	L_2	0.1571	10.86
Head and Neck	L_3	0.1546	10.68
Upper Arm	L_4, L_6	0.1735	11.99
Forearm and Hand	L_5, L_7	0.1714	13.23
Thigh	L_8, L_{10}	0.2399	16.58
Leg and Foot	L_9, L_{11}	0.2505	17.31
1/2 Hip Breadth	L_H	0.0491	3.39
Shoulder Link	L_S	0.0917	6.34
Head Mass Center Link	L_C	0.0108	0.75

The center of mass locations, also obtained from data of references 4 and 5, are calculated as fixed fractions of segment lengths. These fractions and the actual dimensions for a 50th percentile male are presented in table 2.

TABLE 2. SEGMENT CENTER OF MASS LOCATIONS			
Segment	Symbol	Fraction of Segment Length	Dimension for 50th Percentile Male (in.)
Lower Torso	ρ_1	0.4515	5.35
Upper Torso	ρ_2	0.4654	5.05
Head and Neck	ρ_3	0.5670	6.06
Upper Arm	ρ_4, ρ_6	0.4374	5.24
Forearm and Hand	ρ_5, ρ_7	0.6770	8.95
Thigh	ρ_8, ρ_{10}	0.4278	7.09
Leg and Foot	ρ_9, ρ_{11}	0.4264	7.38

2.1.4.2 Body Segment Mass

The masses of body segments are calculated in the same manner · s the lengths, using fractions of total body mass, based on data of references 4 and 5. These fractions and segment weights for a 50th percentile male (161.5-pound weight) are presented in table 3.

TABLE 3. OCCUPANT SEGMENT INERTIAL PROPERTIES

Segment	Fraction of Total Body Mass (M_n/M)	Weight of Segment for 50th Percentile Male* (lb)	Moments of Inertia for 50th Percentile Male ($\text{lb-sec}^2\text{-in.}$)		
			I_x	I_y	I_z
Lower Torso	0.2778	44.86	8.703	4.331	8.703
Upper Torso	0.2264	36.56	3.357	2.623	2.623
Head and Neck	0.0792	12.79	0.311	0.311	0.201
Upper Arm	0.0264	4.26	0.164	0.164	0.0241
Forearm and Hand	0.0214	3.46	0.0241	0.218	0.218
Thigh	0.1001	16.17	0.307	1.270	1.270
Leg and Foot	0.0604	9.75	1.192	1.192	0.120

*Weight, rather than mass is tabulated here, as basic units are generally more meaningful to the reader.

2.1.4.3 Segment Moments of Inertia

Mass moments of inertia, computed from measurements made on eight human cadavers, were reported by Dempster in reference 6. These moments of inertia were all measured with respect to transverse body axes at a convenient point of suspension for each segment and, subsequently, transferred to the segment mass centers. Therefore, from these results the values of I_{y_n} can be extracted, with the exception of the upper torso segment, which will be discussed further below.

The cadavers used in Dempster's experiments were generally of slight build possessing an average weight and stature considerably lower than the 50th percentile. Using the anthropometric data on the United States civilian population contained in reference 7, the weight and stature for the 50th percentile male are 161.5 pounds and 69.1 inches, respectively. For use in the mathematical model, all of Dempster's values were adjusted to account for total body size by multiplication by a factor that is related to the units of mass moment of inertia. In other words, the adjusted values for the 50th percentile male are calculated according to

$$I_{50} = I_D \ (161.5) \ (69.1)^2/W_D S_D{}^2 \tag{52}$$

where I_D is the segment moment of inertia calculated by Dempster for a population of average weight W_D and stature S_D.

Turning to the upper torso, Dempster subdivided this segment into the thorax and two shoulder segments and reported the three moments of inertia separately. For use in the model a composite value of moment of inertia was calculated as described below.

Figure 14(a) shows the location of the center of mass for the thorax anterior to the vertebral body T-9. Figure 14(b) shows the location of the centers of mass of the shoulder segments below rib 3. The lateral location of the shoulder center of mass is unimportant as, for the two shoulder segments together, it will be in the midsagittal plane. Using the superior face of vertebral body T-1 as a reference, the distances to the centers of mass for the thorax and shoulders indicated in figure 14 were obtained by scaling the dimensions on Dempster's sketches. The length given for the thorax link between the superior face of the centrum of vertebra T-1 and the inferior face of the centrum of vertebra T-12 is 15.71 percent of the body stature, or, for the 50th percentile male, 10.86 inches. The vertical distance to the center of mass of the thorax is given as 66.1 percent of the link length or 7.18

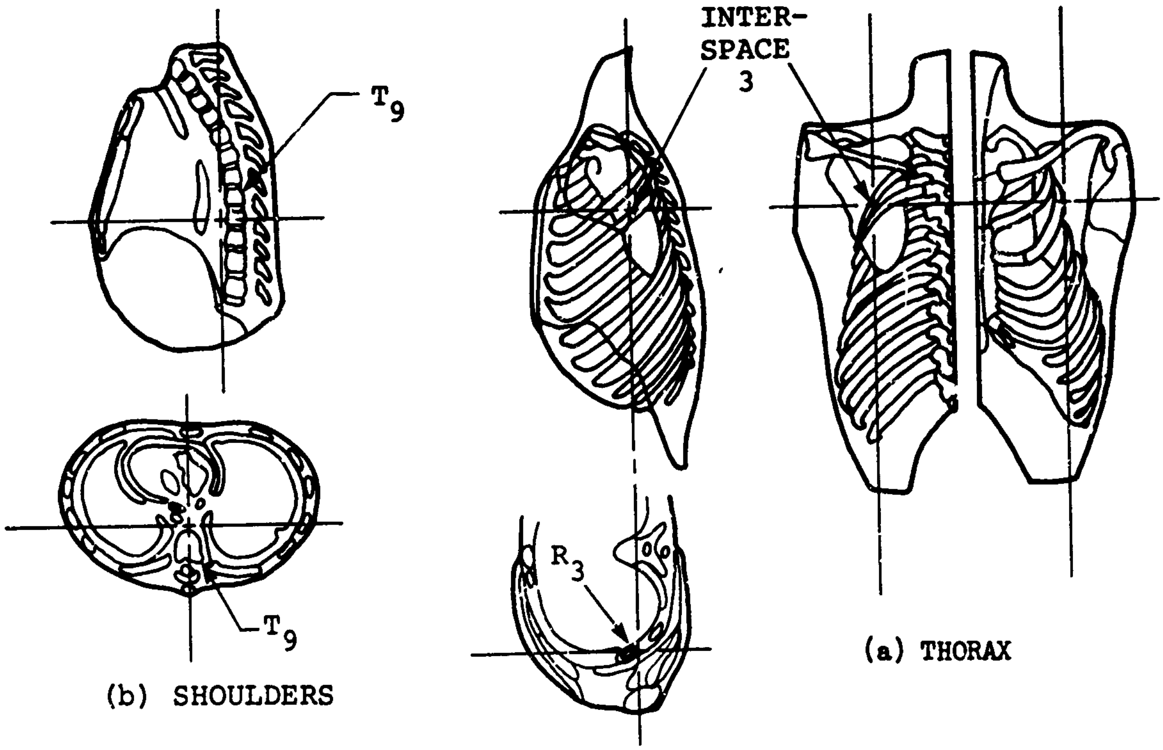


Figure 14. Location of Centers of Mass for Upper Torso Segments (Reference 6).

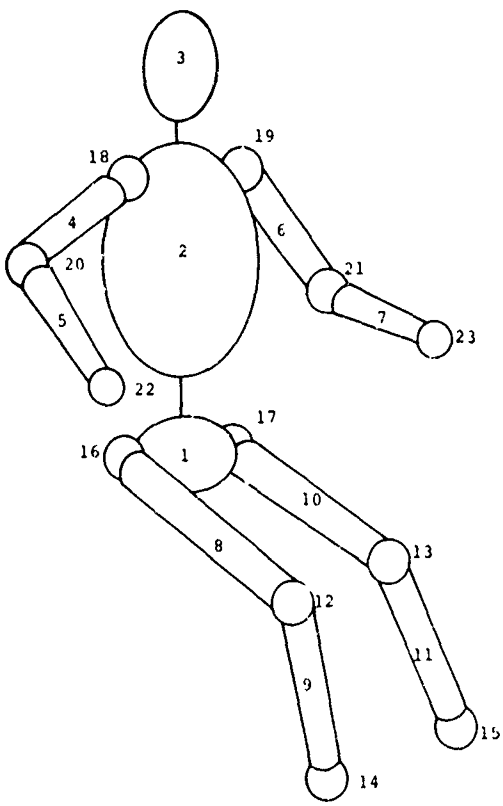
inches for the 50th percentile male. Using the spinous process of T-1, which is visible in figure 14(b) as a reference, the distance from the superior face of T-1 to the mass center of a shoulder segment can be determined to be approximately 28 percent of the link length, or 3.04 inches for the 50th percentile male. The center of mass for the combined segment is then located 5.15 inches or 47.4 percent of the link length below T-1. The moments of inertia can be combined, using the parallel axis theorem, as illustrated in appendix H.

The moments of inertia with respect to the segment x- and z-axes were determined, using approximations to segment geometry similar to the technique described in reference 8. The torso and head segments were approximated by ellipsoids. Assigning appropriate anthropometric dimensions to the ellipsoid axes, the ratios I_{x_n} / I_{y_n} and I_{z_n} / I_{y_n} were calculated for unit mass. These ratios,

multiplied by the I_{y_n} extracted from Dempster's data, gave values of I_{x_n} and I_{z_n} for the torso and head segments. The identical procedure was used for the extremities, except that these segments were approximated by solid circular cylinders. The details of the calculations are presented in appendix H, and the moments of inertia for a 50th percentile male are listed in table 3.

2.1.4.4 Body Contact Surfaces

Twenty-three surfaces are defined on the body for calculation of external forces exerted on the occupant by the seat cushions or restraint system and for prediction of impact between the occupant and the cockpit interior. These surfaces are ellipsoids, cylinders and spheres, as shown in figure 15. The dimensions of these surfaces were obtained from anthropomorphic data on the United States civilian population (reference 7).



SURFACE IDENTIFICATION

1. PELVIS
2. CHEST
3. HEAD
4. RIGHT UPPER ARM
5. RIGHT FOREARM
6. LEFT UPPER ARM
7. LEFT FOREARM
8. RIGHT THIGH
9. RIGHT LOWER LEG
10. LEFT THIGH
11. LEFT LOWER LEG
12. RIGHT KNEE
13. LEFT KNEE
14. RIGHT FOOT
15. LEFT FOOT
16. RIGHT HIP
17. LEFT HIP
18. RIGHT SHOULDER
19. LEFT SHOULDER
20. RIGHT ELBOW
21. LEFT ELBOW
22. RIGHT HAND
23. LEFT HAND

Figure 15. Occupant Contact Surfaces.

The surfaces and the dimensions required for their description are illustrated in detail in figure 16. The values of R_i for a 50th percentile male are listed in table 4, along with their fractions of body stature. The segment lengths L_n and center-of-mass location ρ_n were defined earlier and are presented in tables 1 and 2.

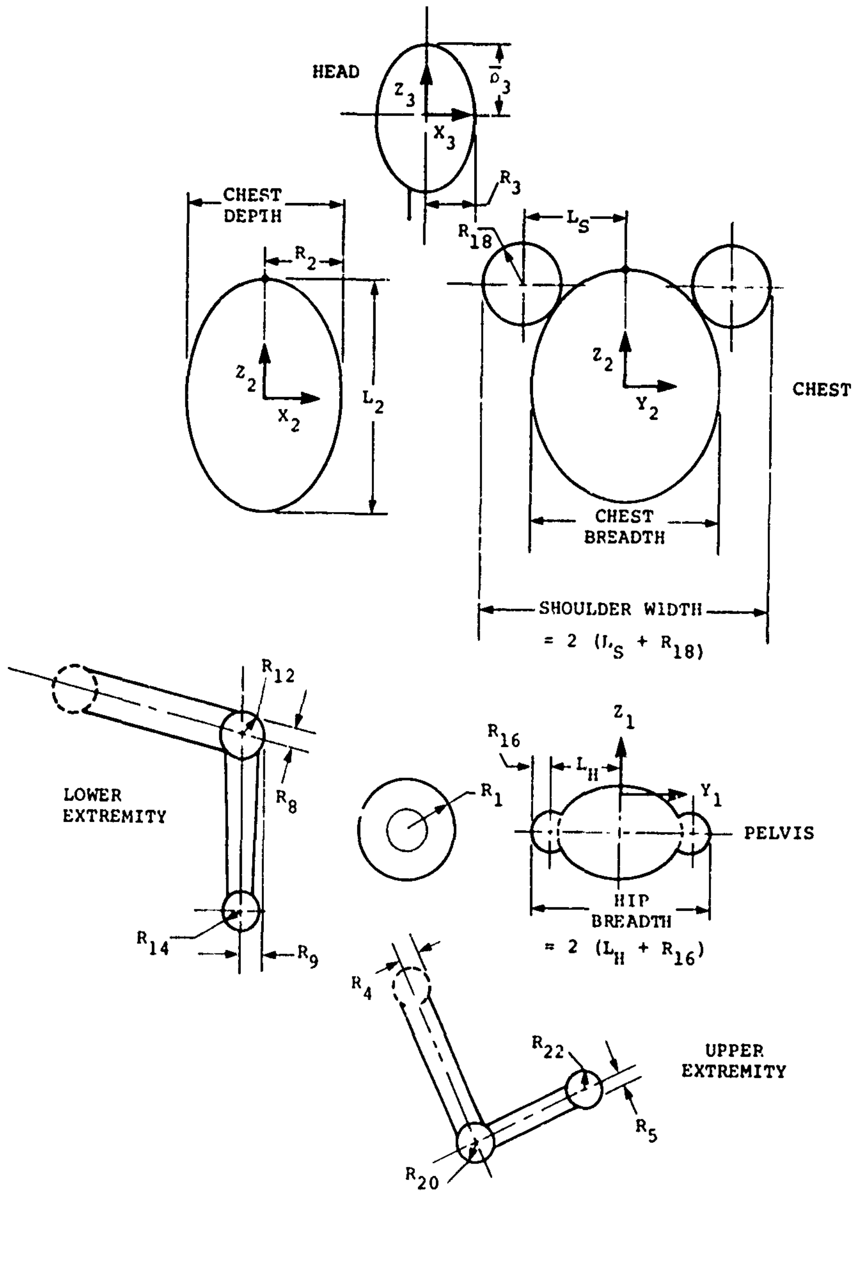


Figure 16. Body Contact Surface Dimensions.

TABLE 4. CONTACT SURFACE DIMENSIONS			
Surface	Symbol	Fraction of Stature (R_i/S)	Actual Dimension for 50th Percentile Male (in.)
Pelvis	R_1	0.0579	4.00
Chest	R_2	0.0651	4.50
Head	R_3	0.0579	4.00
Arm	R_4, R_6	0.0282	1.95
Forearm	R_5, R_7	0.0268	1.85
Thigh	R_8, R_{10}	0.0514	3.55
Leg	R_9, R_{11}	0.0333	2.30
Knee	R_{12}, R_{13}	0.0333	2.30
Foot	R_{14}, R_{15}	0.0250	1.73
Hip	R_{16}, R_{17}	0.0515	3.56
Shoulder	R_{18}, R_{19}	0.0378	2.61
Elbow	R_{20}, R_{21}	0.0268	1.85
Hand	R_{22}, R_{23}	0.0297	2.05

2.1.4.5 Joint Rotation

The results of several studies on the limits of human joint motion have been published. Two of these studies in particular were examined for applicability to the occupant model. First of all, Dempster's (reference 6) data on link lengths and inertial properties were used, as discussed in preceding sections, so it was considered appropriate to include his joint data here. Glanville and Kreezer (reference 9) presented limits of joint motion for both voluntary and forced rotation; their results appear, along with Dempster's, in table 5. Definitions of the various joint motions are illustrated in figure 17. Also included in table 5 are the rotations recommended for anthropomorphic dummy joints by SAE Recommended Practice J963.

TABLE 5. RANGE OF JOINT ROTATION

Body Component Motion	Symbol	Motion Description	Measured Rotation - Deg: Human: Glanville and Kreezer[9] Voluntary	Forced	Dempster[6]	Dummy (SAE J963)
Head - With Respect to Torso	A	Dorsiflextion	61	77	-	60
	B	Ventriflexion	60	7€	-	60
	C	Lateral Flexion	41	63	-	40
	D	Rotation	78	63	-	70
Upper Arm - At Shoulder	E	Abduction (Coronal Plane)	130	137	134	135
	F	Flexion	180	185	188	180
	G	Hyperextension	58	69	61	60
Forearm - At Elbow	H	Flexion	141	146	142	135
Thigh - At Hip	I	Flexion	102	112	113	120
	J	Hyperextension	45	54	-	45
	K	Medial Rotation	-	-	39	50
	L	Lateral Rotation	-	-	34	50
	M	Adduction	-	-	31*	10
	N	Abduction	71	79	53*	50
Lower Leg - At Knee	P	Flexion	125	138	125	135
Long Axis of Torso	Q	Flexion	-	-	-	40
	R	Hyperextension	-	-	-	30
	S	Lateral Flexion	-	-	-	35
	T	Rotation	-	-	-	35

*Transverse plane.

All of the rotations possible in the mathematical model are included in table 5 and figure 17, but some are, naturally, more important than others in determining permissible ranges of motion for the model. For the head, ventriflexion (B) is certainly the most important component of motion for frontal impact. Dorsiflexion (A) may also be important for frontal impact, but the angles reported are sufficiently close to those for ventriflexion to be considered the same. Lateral flexion (C) is certainly less important since a pure lateral impact of an aircraft would be rare indeed, and rotation (D) will have an insignificant effect on model response. Therefore, the limiting rotation β_{S_i} (see section 2.1.1.2) for the neck joint (i = 2) has been taken as the

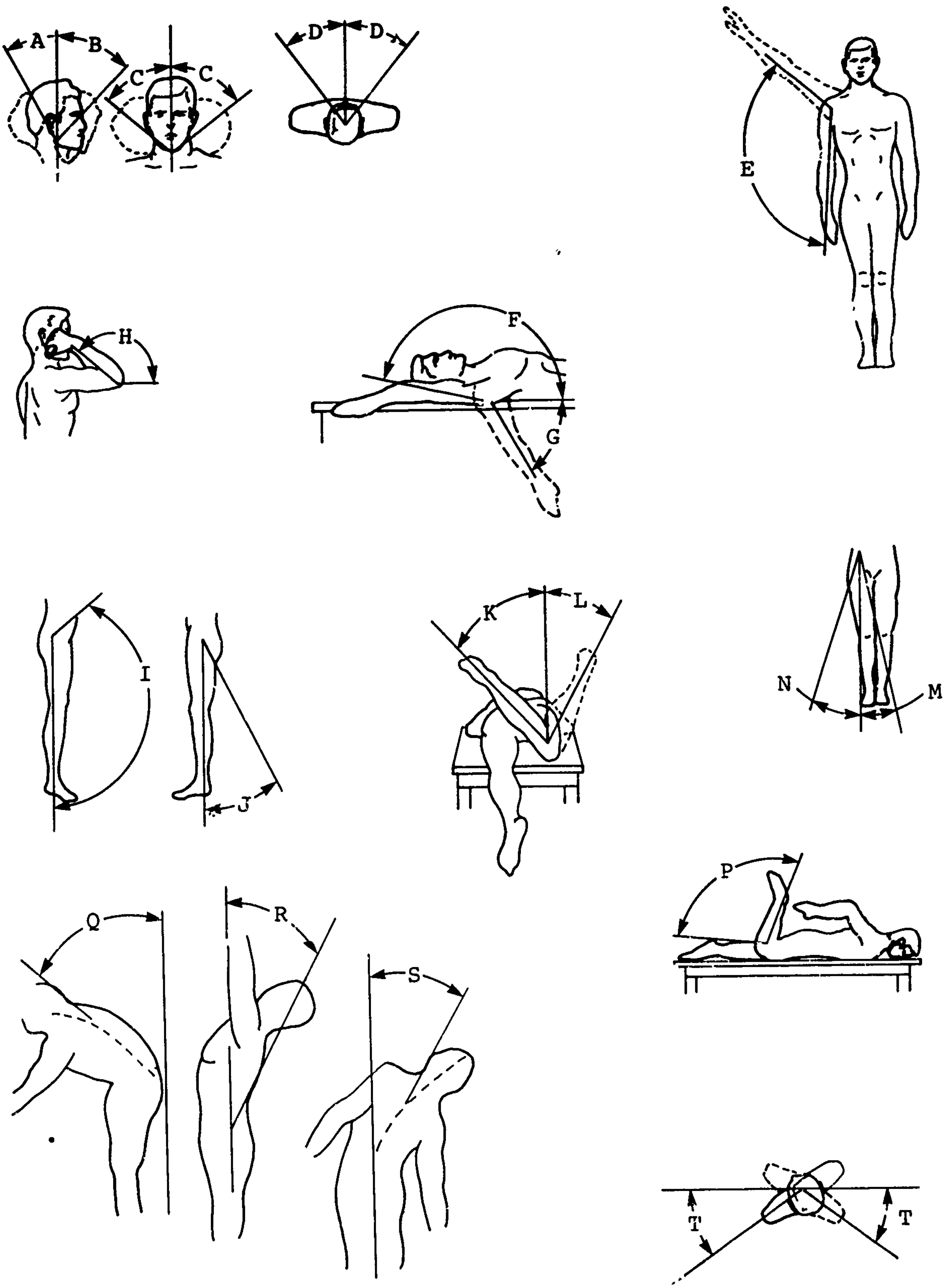


Figure 17. Motion Diagrams.

limit for voluntary ventriflexion, or 60 degrees. The angle can be found in table 6, along with the limiting angles for the other body joints. For all of the other angles, flexion is the most significant component for the type of motion that can usually be expected to take place in a crash environment. Therefore, the limiting angles were all taken as the limits for voluntary flexion. Note that, for the hip joint, the reference position of the body used in the mathematical model includes 90-degree flexion. Therefore, this amount has been subtracted from the angle reported in table 5, which is defined relative to the standard anatomical reference position. Since the seated position appears to aid in flexion of the hip joint, the largest angle in the table, the one given by SAE J963, was used in determining β_{S_7} which is thus given by $\beta_{S_7} = 120° - 90° = 30°$.

TABLE 6. JOINT LIMITING ANGLES

Joint	Location	Angle - β_{S_i} (deg)
1	Back	40
2	Neck	60
3, 5	Shoulder	180
4, 6	Elbow	142
7, 9	Hip	30
8, 10	Knee	125

2.2 SEAT MODEL

In order to make maximum use of commonality and to minimize input complexity, the seat model is divided into two major components. The first, composed of the seat pan and back, retains the same configuration regardless of the particular type of seat being analyzed. The supporting structure, on the other hand, can

vary widely in its overall geometry and number of elements. The characteristics of each of the components of the seat model are discussed below with attention being focused first on the common part of the structure, which is actually made up of three types of elements, as shown in figure 18.

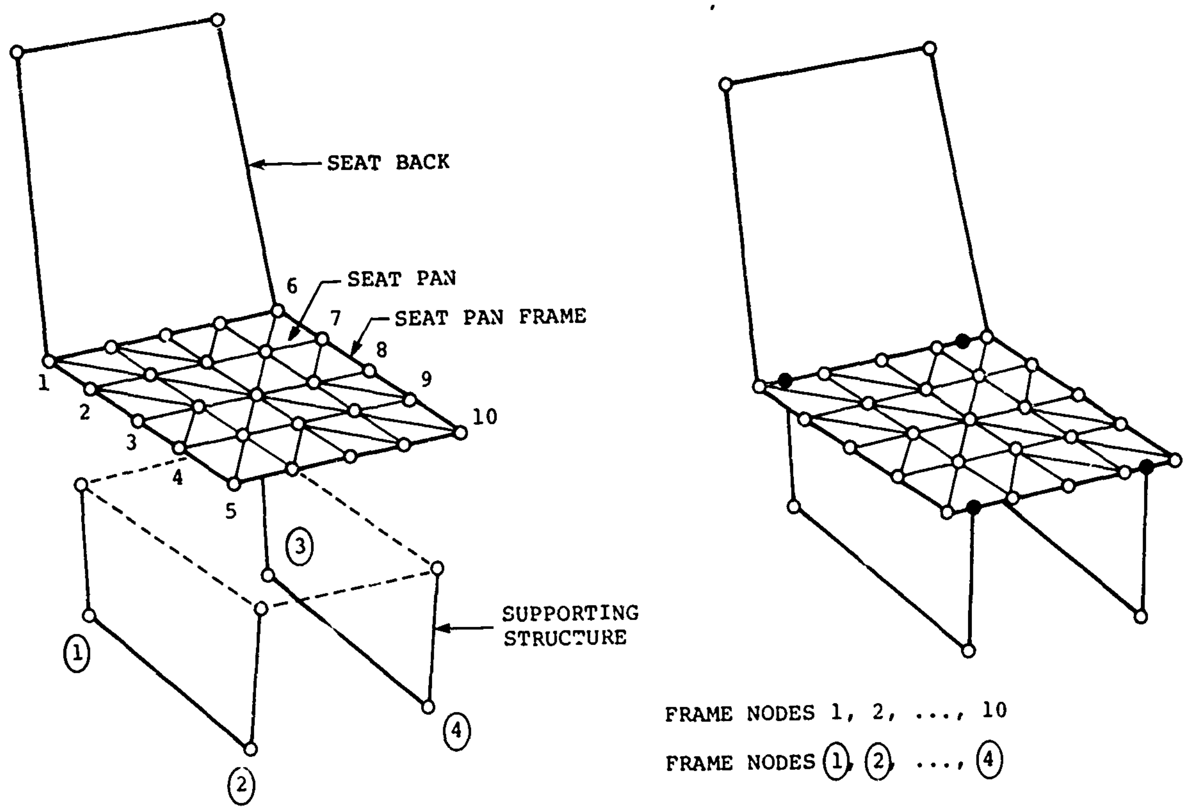


Figure 18. Seat Model Components.

The seat pan is composed of triangular membrane elements. The seat cushion load is distributed parabolically over the inner nine nodes. The sample input data include fictitious mechanical properties for these elements that have been determined to best simulate a fabric or spring-supported seat pan. A sheet-metal seat pan would be represented by using actual material properties as input data.

A seat pan frame formed of 16 beam elements transmits the seat pan loads to the supporting structure. Also applied to this frame are loads from the seat back, the occupant's thighs, and the lap belt, should the user wish the belt attached to the seat.

The seat back is made up of three simple beams of uniform cross section. The back cushion loads are assumed to be distributed along the sides of the seat back and are thus transmitted to the seat pan frame. Should the shoulder harness be attached to the seat, its load is applied at the midpoint of the transverse beam that forms the top of the seat back. Bending of these beams can alter the seat geometry during the crash event, and bending failure at the connection to the seat pan frame can be predicted, based on an input value of ultimate stress for the frame material.

2.2.1 Seat Analysis

The behavior of a typical occupied seat in an impact situation is characterized by three distinct, though overlapping phases. Initially the seat pan is only slightly deformed and comparatively flexible. The resistance of such a "flat" membrane to normal displacement arises primarily from its current state of stress and current geometry. Once the seat pan is sufficiently deformed to be of comparable stiffness to the frame, the second phase of behavior is pursued. Herein the framework and the pan continue to deform elastically with neither contribution necessarily dominant. The flexure of the supporting frame acts to soften the pan stiffness while continued stretching of the pan counterbalances this influence. In the final phase, frame components may become plasticized, thereby becoming incapable of carrying greater load. Plastic hinges are introduced, causing redistribution of the loads. Redistribution continues until sufficient hinges are formed to allow the structure to deform as a kinematic mechanism, at which point collapse occurs.

The present analysis accounts for the above described behavior although no attempt is made to discretize the separate phases. All phenomena are considered throughout the analysis,

thus providing gradual transition from initiation of deformation to termination, possibly by total collapse.

The data handling for the seat analysis is designed to efficiently utilize limited computer storage capacity by not allowing large matrices, generated for various components, to reside in core simultaneously. The general flow, described in more detail subsequently, involves first forming the structural matrix describing the particular supporting structure. This matrix is modified during formulation to account for any plastic hinges or floor connection failure. Once formed, the matrix is copied to auxiliary storage and numerically collapsed to the seat pan frame intersection nodes. The seat pan stiffness matrix is then created, added to the leg frame residue, and the resulting combined matrix saved on auxiliary storage until further collapsed to the seat pan intersection.

Similarly, the seat pan matrix is formed and solved for the increments of displacement. Frame, leg, and bar matrices are retrieved and used to calculate bar forces and floor reactions. Updated displacement data are stored to define current position in the subsequent call to seat routine.

Discussions of analysis techniques performed in the program, which are applicable to one or more of the seat types, are presented below.

2.2.1.1 Bar Stiffness

A straight bar of constant cross section, as shown in figure 19, is taken as the first element. For small elastic deformation the customary approximations used in finite-element developments are unnecessary because the force-displacement relationships are well known. The six components of displacement at each end are related to the corresponding generalized forces by the matrix expression

$$\{P\} = \{K\} \{\rho\} \tag{53}$$

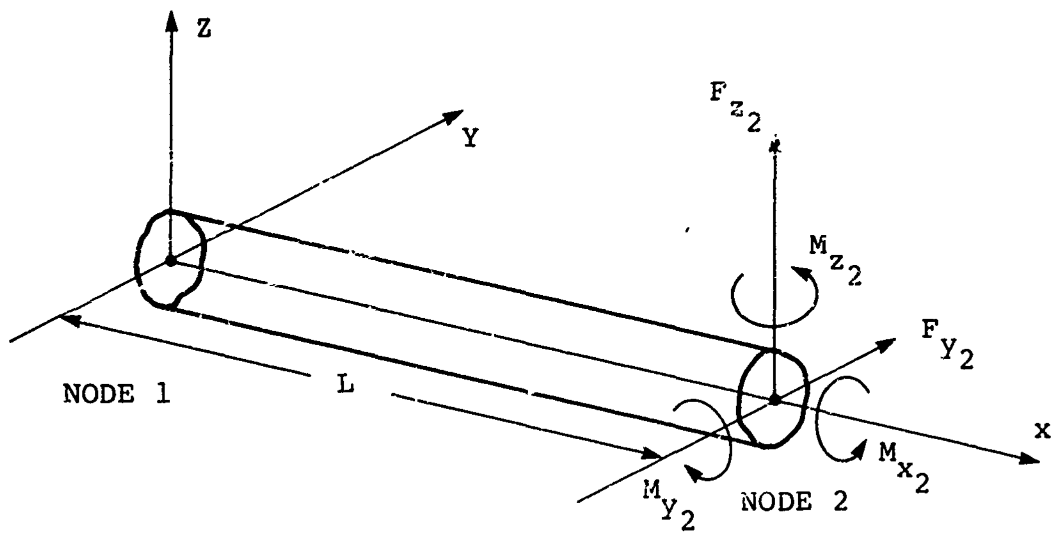


Figure 19. Space Frame Element in Elemental Coordinate System.

where $\{P\}$ = the 12 x 1 column vector of nodal forces

$\{\rho\}$ = the 12 x 1 column vector of nodal displacements

$\{K\}$ = the 12 x 12 stiffness matrix

In expanded notation the equivalent expression is given by equation (54)

where A = area of cross section

E = modulus of elasticity

I = moment of inertia

G = modulus of rigidity

J = torsion constant

L = length of member

M_{x_i}, (M_{y_i}), (M_{z_i}) = moment about X, (Y), (Z) at node i

F_{x_i}, (F_{y_i}), (F_{z_i}) = force in X, (Y), (Z) direction at node i

θ_{x_i}, (θ_{y_i}), (θ_{z_i}) = rotation about X, (Y), (Z) axis at node i

u_i, (v_i), (w_i) = X, (Y), (Z) component of idsplacement at node i

$$
\begin{Bmatrix} F_{x_1} \\ F_{y_1} \\ F_{z_1} \\ M_{x_1} \\ M_{y_1} \\ M_{z_1} \\ F_{x_2} \\ F_{y_2} \\ F_{z_2} \\ M_{x_2} \\ M_{y_2} \\ M_{z_2} \end{Bmatrix}
=
\begin{bmatrix}
\frac{AE}{L} & & & & & & -\frac{AE}{L} & & & & & \\
 & \frac{12EI_z}{L^3} & & & & \frac{6EI_z}{L^2} & & -\frac{12EI_z}{L^3} & & & & \frac{6EI_z}{L^2} \\
 & & \frac{12EI_y}{L^3} & & -\frac{6EI_y}{L^2} & & & & -\frac{12EI_y}{L^3} & & -\frac{6EI_y}{L^2} & \\
 & & & \frac{GJ}{L} & & & & & & -\frac{GJ}{L} & & \\
 & & & & \frac{4EI_y}{L} & & & & \frac{6EI_y}{L^2} & & \frac{2EI_y}{L} & \\
 & & & & & \frac{4EI_z}{L} & & -\frac{6EI_z}{L^2} & & & & \frac{2EI_z}{L} \\
 & & & & & & \frac{AE}{L} & & & & & \\
 & & \text{SYMMETRIC} & & & & & \frac{12EI_z}{L^3} & & & & -\frac{6EI_z}{L^2} \\
 & & & & & & & & \frac{12EI_y}{L^3} & & \frac{6EI_y}{L^2} & \\
 & & & & & & & & & \frac{GJ}{L} & & \\
 & & & & & & & & & & \frac{4EI_y}{L} & \\
 & & & & & & & & & & & \frac{4EI_z}{L}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \theta_{x_1} \\ \theta_{y_1} \\ \theta_{z_1} \\ u_2 \\ v_2 \\ w_2 \\ \theta_{x_2} \\ \theta_{y_2} \\ \theta_{z_2} \end{Bmatrix}
\qquad (54)
$$

2.2.1.2 Rigid Body Transformation

In general, actual seat supporting structure attachments might only rarely coincide with natural integral division points of the seat pan frame. A rigid body transformation technique is employed to accommodate this "mismatch", without increasing the problem size, by the obvious technique of adding nodes. By this method nodal displacements and bar end displacements are related through kinematics, resulting in a local approximation of stresses, but providing full equilibrium and continuity.

Figure 20 illustrates the geometry and nomenclature. The bar end B is eccentric from the node N by the distances (e_x, e_y, e_z)

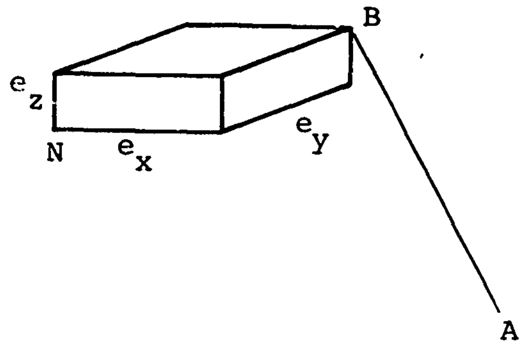


Figure 20. Geometry of Rigid-body Transformation.

The bar end displacements are related, in a purely kinematic sense, to the nodal displacements by the matrix equation

$$\{e_B\} = [A]\ \{e_N\} \tag{55}$$

and $\{e_{B}(N)\}$ represents the 6x1 vector of displacement at B(N) and [A] is the rigid body relation defined by

$$[A] = \left[\begin{array}{c|ccc} & o & e_z & e_y \\ I & e_z & o & e_x \\ & e_y & e_x & o \\ \hline O & & I & \end{array}\right] \tag{56}$$

The transpose of [A] represents the same rigid body transformation applied to the corresponding force components.

$$F_B = A^T F_N \tag{57}$$

The transformed stiffness matrix is then given by the congruent transformation

$$K' = T^T K T \tag{58}$$

where [T] is formed from [A]

$$T_{12x12} = \left[\begin{array}{c|c} A & O \\ \hline O & I \end{array}\right] \tag{59}$$

2.2.1.3 Matrix Reduction

In order to use available computer storage economically it is necessary to reduce larger order matrixes for components to their influence on freedoms common to other components. The general technique explained subsequently performs this operation. The subscript i refers to the variables to be eliminated while j refers to those retained. The combined (i and j) matrix is partitioned into the indicated submatrixes.

$$K = \left[\begin{array}{c|c} K_{ii} & K_{ij} \\ \hline K_{ji} & K_{jj} \end{array}\right] \tag{60}$$

The corresponding equilibrium equations are

$$K_{ii}\, r_i + K_{ij}\, r_j = R_i \tag{61}$$

$$K_{ij}\ r_i + K_{jj}\ r_j = R_j \tag{62}$$

Solution of equation (61) gives

$$r_i = K_{ii}^{-1}\ [R_i - K_{ij}\ r_j] \tag{63}$$

Substitution of equation (63) into equation (62) gives

$$[K_{jj} - K_{ji}\ K_{ii}^{-1}\ K_{ij}]\ r_j = R_j - K_{ij}\ K_{ii}^{-1}\ R_i \tag{64}$$

The bracketed quantity on the left side of the equation is the modified or reduced stiffness matrix; a modified load vector is represented on the right side.

2.2.1.4 Plastic Modification

A plastic hinge is introduced at any point in the analysis when the bending moment at a bar end reaches a predetermined limiting value. This requires a modification of the bar stiffness matrix to remove the capability to resist further moment. The modification is performed as follows.

The stiffness matrix presented previously represents the self-equilibrating set of forces at the bar ends induced by the end displacements. The physical condition that one of these components be zero is accomplished by the matrix reduction of the previous section, where in this application K refers to the 12x12 bar stiffness matrix, i is the component of zero moment, and j the remaining nonplasticized degrees of freedom.

The ii term is replaced by a small number for convenience in computation to prevent an apparent mechanism from developing.

2.2.1.5 Triangular (Membrane) Element Stiffness

<u>Elastic Stiffness</u> - The triangular panel of figure 21 is considered. The bending stiffness is ignored and the state of membrane stress is assumed constant throughout the element.

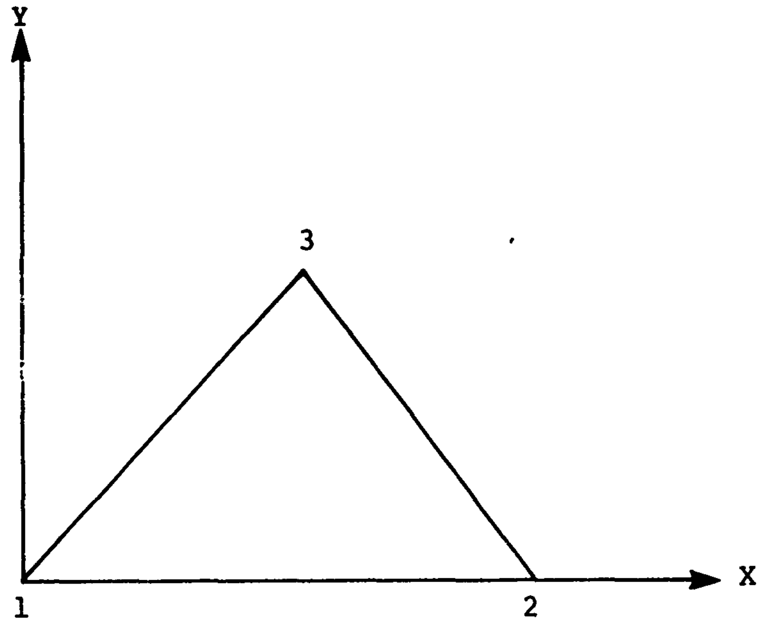


Figure 21. Triangular Panel Element.

Nodal points are the three corners. The relationship between representative nodal force and in-plane nodal displacements (or stiffness matrix) is

$$
[K] = \lambda_1 \begin{bmatrix}
y_3^2 + \lambda_2 \lambda_3^2 & & & & & \\
- y_3^2 & - \lambda_2\lambda_3 x_3 y_3^2 + \lambda_2 x_3^2 & & \text{- Symmetric -} & & \\
\lambda_2\lambda_3 x_2 & - \lambda_2 x_2 \lambda_3 & \lambda_2 x_2^2 & & & \\
- y_3 (\lambda_2 + v) & y_3 (\lambda_2 x_3 + v\lambda_3) & - \lambda_2 x_2 y_3 & \lambda_2 y_3^2 + \lambda_3^2 & & \\
y_3 (\lambda_2\lambda_3 + v x_3) & - x_3 y_3 (\lambda_2 + v) & \lambda_2 x_2 v_3 & - \lambda_2 y_3^2 - x_3\lambda_3 & \lambda_2 y_3^2 + x_3^2 & \\
- v x_2 y_3 & v x_2 y_3 & 0 & x_2\lambda_3 & - x_2 x_3 & x_2^2
\end{bmatrix}
$$

$$
\lambda_1 = \frac{Et}{2x_2 y_3 (1 - v^2)}, \quad \lambda_2 = \frac{1 - v}{2}, \quad \lambda_3 = x_3 - x_2 \qquad (65)
$$

__Geometric Stiffness__ - The geometric stiffness of a triangular panel is presented subsequently. Physically, it represents the forces required to displace (rotate) a stretched membrane normal to its plane.

The stiffness is expressed in matrix form as

$$k_g = t \int \{w_x^T, w_y^T\} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{Bmatrix} w_x \\ \\ w_y \end{Bmatrix} dA \qquad (66)$$

where σ_x (σ_y) = the direct stress in the x(y) direction

τ_{xy} = the membrane shearing stress

t = plate thickness

A = area

w = displacement normal to the plane of the plate

and x, y are the in-plane position ordinates; w_x, w_y are slopes in the subscripted directions. With the assumption that the average slopes are sufficiently representative of the detailed deformation, the derivatives w_x, w_y are expressed directly in terms of the corner displacements normal to the plate surface.

2.2.1.6 Coordinate Transformations

The elemental stiffness matrix formulations are most conveniently carried out in the natural local coordinates of the element.

The congruent rigid body transformation, in reality, is only a rather special case of generalized coordinate transformation. In the general case, wherein T represents the relation between coordinate systems, the congruent transform becomes

$$\overline{K} = T^T K T \qquad (67)$$

For the particular case of axis rotation

$$T = \begin{bmatrix} t & 0 \\ & \\ 0 & t \end{bmatrix}, \tag{68}$$

and

$$t = \begin{bmatrix} \ell_x & \ell_y & \ell_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \tag{69}$$

where ℓ_x, ℓ_y, ℓ_z are the direction cosine of global X in the the local xyz system

m_x, m_y, m_z are the direction cosines of global Y

n_x, n_y, n_z are the direction cosines of global Z

2.2.1.7 Master Matrix Formulation

Elemental matrices are combined by the direct stiffness method into a "master" matrix for the seat structure. This procedure can be represented by the matrix equation

$$K = [B]^T \; [K_{elt}] \; [B] \tag{70}$$

where K_{elt} contains the element stiffness matrices as submatrices on the main diagonal and zeros elsewhere.

$$K_{elt} = \begin{bmatrix} k_1 & & & & & \\ & k_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix} \tag{71}$$

and B is a Boolean transformation matrix relating the local and master degrees of freedom.

Computationally, this operation never actually materializes because it would physically amount to adding the local stiffness matrices into the global system in proper order. This is best accomplished one element at a time by direct addition.

2.2.2 Seat Types

The configuration of the supporting structure can be varied in order to describe a particular seat. The only information required for input involves seat dimensions and material properties. The simplicity of the input data is possible because the geometry of four different types of seat configurations is included in the program. These four seat configurations, which are capable of simulating nearly every seat found in light aircraft, are defined below and illustrated in figure 22.

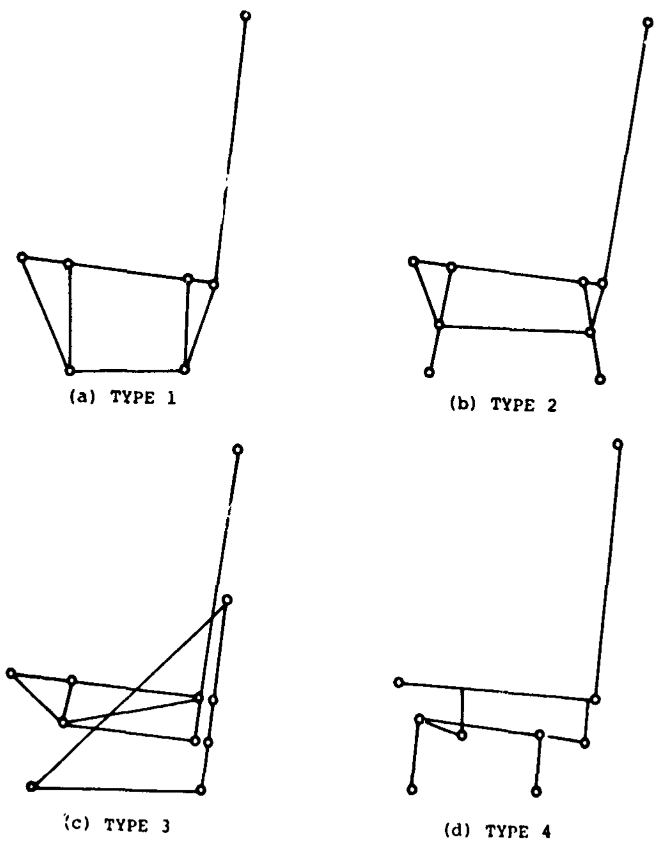


Figure 22. Seat Types.

Type 1. The first seat type is the most general and is representative of the greatest number of seat configurations. An example is shown in figure 22(a). The supporting structure which need not be symmetric, consists of an arbitrary number of beam elements that connect up to six nodes on the aircraft floor with any of the 10 nodes on the outer edges of the seat pan frame, as well as with each other. As an example, the type 1 seat shown in figure 18 has a supporting structure consisting of six bars that connect four floor nodes with four frame nodes and with each other.

The primary restriction on the response of the type 1 seat is that all frame nodes are fixed to resist rotation, as in the case of welded joints. However, at any point in the analysis, should the bending moment at a bar end reach a predetermined limiting value, a plastic hinge is introduced. The limiting moments are based on input values of yield stress and provide a means for simulating bar failure.

Type 2. The second seat type has up to 14 beam elements in the supporting structure. Bars may connect several pinned-type joints on the seat legs, but the number of floor nodes is restricted to four. An example is shown in figure 22(b).

Type 3. This seat configuration has a substantially more complex structure than the previous two. As shown in figure 22(c), the seat is vertically adjustable with adjustment provided by motion of the seat-supporting frame along relatively rigid guide tubes.

Type 4. This seat is another vertically adjustable configuration with adjustment provided by rotation of crank mechanisms, as shown in figure 22(d).

3.0 SIMULATION COMPUTER PROGRAM

The digital computer program based on the occupant and seat models described in section 2.0 is called Seat Occupant Model-Light Aircraft (SOM-LA). It has been written entirely in FORTRAN IV to ensure a high degree of compatibility with various digital computer systems. During development the program has been run on both UNIVAC 1108 and CDC 6600 computer systems.

The elements of the program can be considered in terms of three general operations:

- Input and Initialization
- Solution
- Output

which are summarized below and discussed in detail in the sections following. The general flow of the program is illustrated in figure 23. Input data describing the occupant and crash conditions are read first. If the user requests output of the prediction of impact between the occupant and the aircraft interior, the coordinates defining the cockpit surfaces are read. Finally, the seat data, either simple dimensions describing a rigid seat model or detailed design data on a flexible seat, are provided. Based on the input data, the values of constants, such as occupant dimensions and properties are calculated, and the initial position of the occupant is determined.

The solution loop is entered for the first time with the aircraft initial velocity and the occupant initial position. At each subsequent entrance to the loop the current aircraft displacement, velocity, and acceleration components are calculated. The equations of motion for the occupant are set up and solved. If a flexible seat is being used, the forces applied to the seat, such as the cushion forces, are provided to the seat routines for computation of seat displacements. At time increments equal to a predetermined print interval, the output variables requested by the user are stored for printing after completion of the solution.

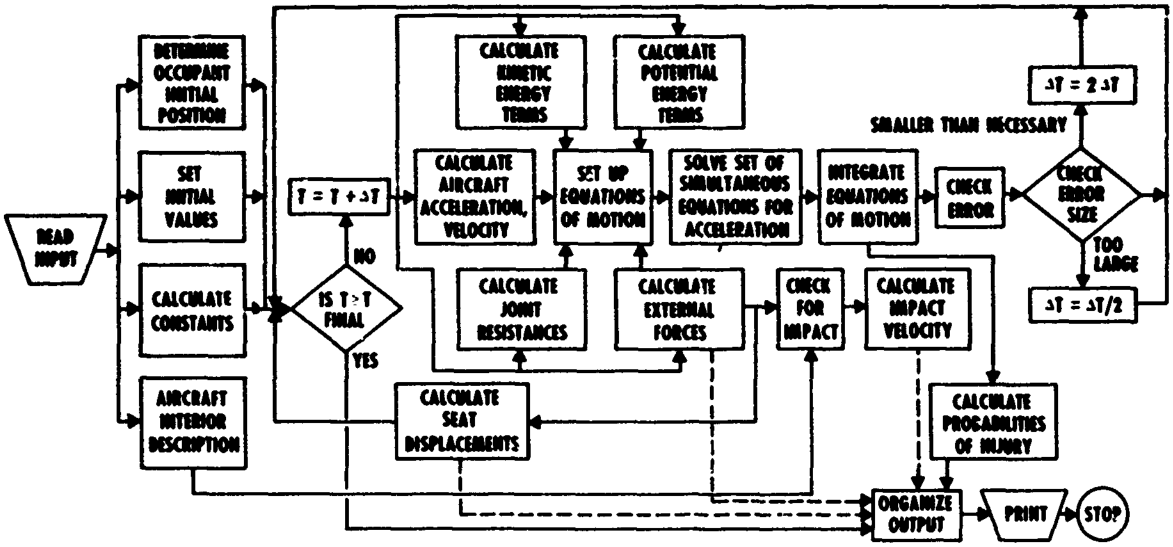


Figure 23. Program Flow Chart.

3.1 PROGRAM INPUT AND INITIALIZATION

Input data are read by the program in the following seven blocks:

1. Simulation control information
2. Occupant description
3. Restraint system description
4. Cushion properties
5. Cockpit description
6. Crash conditions
7. Seat design information

3.1.1 Simulation Control Information

The first block of data contains the information required for controlling execution of the program. The initial time step for integration of the equations of motion, the total length of the simulation, the number of cases to be run, the system of units (SI or English), and identification of the desired output are provided here.

3.1.2 Occupant Description

Because it has been assumed that the principal user of this program is interested primarily in the seat or restraint system, a minimum of information is required to describe the occupant. Data include a parameter that defines the occupant type - human or dummy - and the size of that occupant. For a human occupant the stature and total weight are required; for a dummy, the percentile (95th male, 50th male, or 5th female).

3.1.2.1 Occupant Properties

The dimensions and inertial properties of the 11 body segments are determined within the program, as discussed in section 2.1.4.

3.1.2.2 Occupant Initial Position

The initial position of the aircraft occupant is computed from the input parameters shown in figure 24. It is assumed that the occupant is seated symmetrically with respect to the aircraft X_A - Z_A plane or, equivalently, that the segment fixed y_n-axes are all parallel to the Y_A-axis. The angular coordinates γ_i (i = 1, 2, 3, 4) define the rotation of segments 1-4 relative to the Z_A axis and, because of the symmetry condition, segment 6 is parallel to segment 4. The angle α describes the position of the forearms relative to the upper arms, and is the initial value of α_5 and α_7. The distance X_H is the initial X-coordinate of the heels (the inferior ends of segments 9 and 11). The procedure described below consists of seating the occupant in such a position that static equilibrium is achieved among the forces exerted by the seat cushion, floor, and either the restraint system or the back cushion.

The first step in determining the initial position involves calculating the Euler angles for the torso, head, and arm segments, since this procedure does not require consideration of the forces due to the cushions and floor. Because the input parameters illustrated in figure 24 define the position of the occupant in

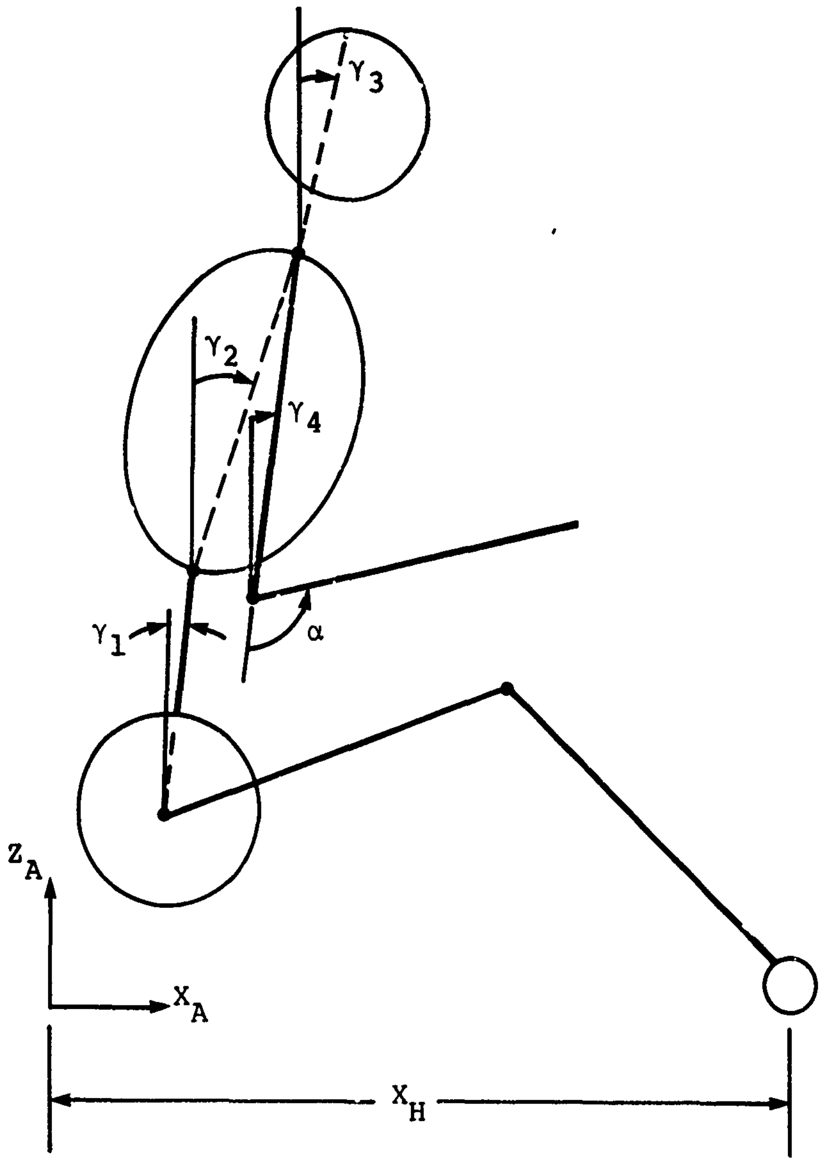


Figure 24. Initial Position Input Parameters.

the aircraft coordinate system, the orientation of the aircraft must be described in the inertial system. For an aircraft in level flight with zero pitch, roll, and yaw it is assumed that the aircraft coordinate axes (X_A, Y_A, Z_A) are parallel to the fixed coordinate axes (X, Y, Z) at the initial time (t = 0). A

general orientation of the aircraft reference frame is obtained by the same sequence of rotations defined in section 2.1.1.1 for the body segments. Defining the rotations

ψ_A: Yaw

θ_A: Pitch

ϕ_A: Roll

the orientation of the aircraft relative to the inertial system is described by the coordinate transformation

$$\begin{matrix} X \\ Y \\ Z \end{matrix} = A \begin{matrix} X_A \\ Y_A \\ Z_A \end{matrix} \tag{72}$$

where the elements of [A] are

$$A_{11} = \cos \psi_A \cos \theta_A$$

$$A_{12} = \cos \psi_A \sin \theta_A \sin \phi_A - \sin \psi_A \cos \phi_A$$

$$A_{13} = \cos \psi_A \sin \theta_A \cos \phi_A + \sin \psi_A \sin \phi_A$$

$$A_{21} = \sin \psi_A \cos \theta_A$$

$$A_{22} = \sin \psi_A \sin \theta_A \sin \phi_A + \cos \psi_A \cos \phi_A$$

$$A_{23} = \sin \psi_A \sin \theta_A \cos \phi_A - \cos \psi_A \sin \phi_A$$

$$A_{31} = -\sin \theta_A$$

$$A_{32} = \cos \theta_A \sin \phi_A$$

$$A_{33} = \cos \theta_A \cos \phi_A \tag{73}$$

The rotation of body segment n relative to the aircraft, remembering that the symmetry condition requires that y_n is parallel to Y_A, is described by

$$\begin{Bmatrix} X_{A_n} \\ Y_{A_n} \\ Z_{A_n} \end{Bmatrix} = \begin{bmatrix} \cos\gamma_n & 0 & \sin\gamma_n \\ 0 & 1 & 0 \\ -\sin\gamma_n & 0 & \cos\gamma_n \end{bmatrix} \begin{Bmatrix} x_n \\ y_n \\ z_n \end{Bmatrix} \tag{74}$$

Combining equations (72) and (74) results in the angular relationship between the local coordinate system of segment n and the inertial system expressed by the following transformation, which is a function of the input γ_n and the aircraft pitch, roll, and yaw:

$$\begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = \begin{bmatrix} B^n \end{bmatrix} \begin{Bmatrix} x_n \\ y_n \\ z_n \end{Bmatrix} \tag{75}$$

where $[B^n]$ is given by

$$\begin{bmatrix} B^n \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \cos\gamma_n & 0 & \sin\gamma_n \\ 0 & 1 & 0 \\ -\sin\gamma_n & 0 & \cos\gamma_n \end{bmatrix} \tag{76}$$

so that its elements are

$$B_{11}{}^n = A_{11}\cos\gamma_n - A_{13}\sin\gamma_n$$

$$B_{12}{}^n = A_{12}$$

$$B_{13}^{\ n} = A_{11} \sin \gamma_n + A_{13} \cos \gamma_n$$

$$B_{21}^{\ n} = A_{21} \cos \gamma_n - A_{23} \sin \gamma_n$$

$$B_{22}^{\ n} = A_{22}$$

$$B_{23}^{\ n} = A_{21} \sin \gamma_n + A_{23} \cos \gamma_n$$

$$B_{31}^{\ n} = A_{31} \cos \gamma_n - A_{33} \sin \gamma_n$$

$$B_{32}^{\ n} = A_{32}$$

$$B_{33}^{\ n} = A_{31} \sin \gamma_n + A_{33} \cos \gamma_n \tag{77}$$

Comparison of equation (75) with equation (1) points out that the transformation matrices $[T^n]$ and $[B^n]$ are equivalent. Because $[T^n]$ is a function of the Euler angles for segment n, equating the elements of $[T^n]$ and $[B^n]$ through

$$[T^n] = [B^n] \qquad n = 1, 2, 3, 4, 6, 8, 10 \tag{78}$$

permits calculation of the initial values of the generalized coordinates from input parameters γ_A and ψ_A, θ_A, and ϕ_A. The procedure as used in Program SOM-LA is outlined below.

First θ_n is determined as follows:

$$T_{31}^n = B_{31}^n \text{ or } -\sin \theta_n = B_{31}^n$$

gives

$$\theta_n = \sin^{-1} (-B_{31}^n) \tag{79}$$

The cosine is then found by

$$\cos \theta_n = \cos [\sin^{-1} (-B_{31}^n)]$$

so that ψ_n can be determined:

$$T_{11}^n = B_{11}^n \text{ or } \cos\psi_n \cos\theta_n = B_{11}^n$$

gives

$$\psi_n = \cos^{-1}(B_{11}^n/\cos\theta_n) \tag{80}$$

and, for determination of ϕ_n,

$$T_{33}^n = B_{33}^n \text{ or } \cos\theta_n \cos\phi_n = B_{33}^n$$

gives

$$\phi_n = \cos^{-1}(B_{33}^n/\cos\theta_n) \tag{81}$$

Equations (79) through (81) are used for segments 1, 2, 3, and 4; the symmetry requirement provides the Euler angles for segment 6, which are equal to those for segment 4. At this point the generalized coordinates q_4 through q_{20} have been determined. The next step involves seating the occupant and calculating X_1, Y_1, and Z_1 (q_1, q_2, and q_3) from static equilibrium.

Because the problem of seating the occupant is statically indeterminate, certain simplifying assumptions are made. The first assumption, which is approximately correct for typical seating positions, is that 15 percent of the occupant's weight is supported by the floor. In other words, 85 percent is supported by the seat cushion and, depending on the aircraft attitude, the restraint system or the back cushion.

A first approximation to the initial position is made for the assumption of level flight ($\theta_A = \psi_A = \phi_A = 0$). The cushion forces act on the body as shown in figure 25, where it is assumed that fifteen percent of the occupant weight is supported by the floor, as discussed in the preceding paragraph. Summing forces gives

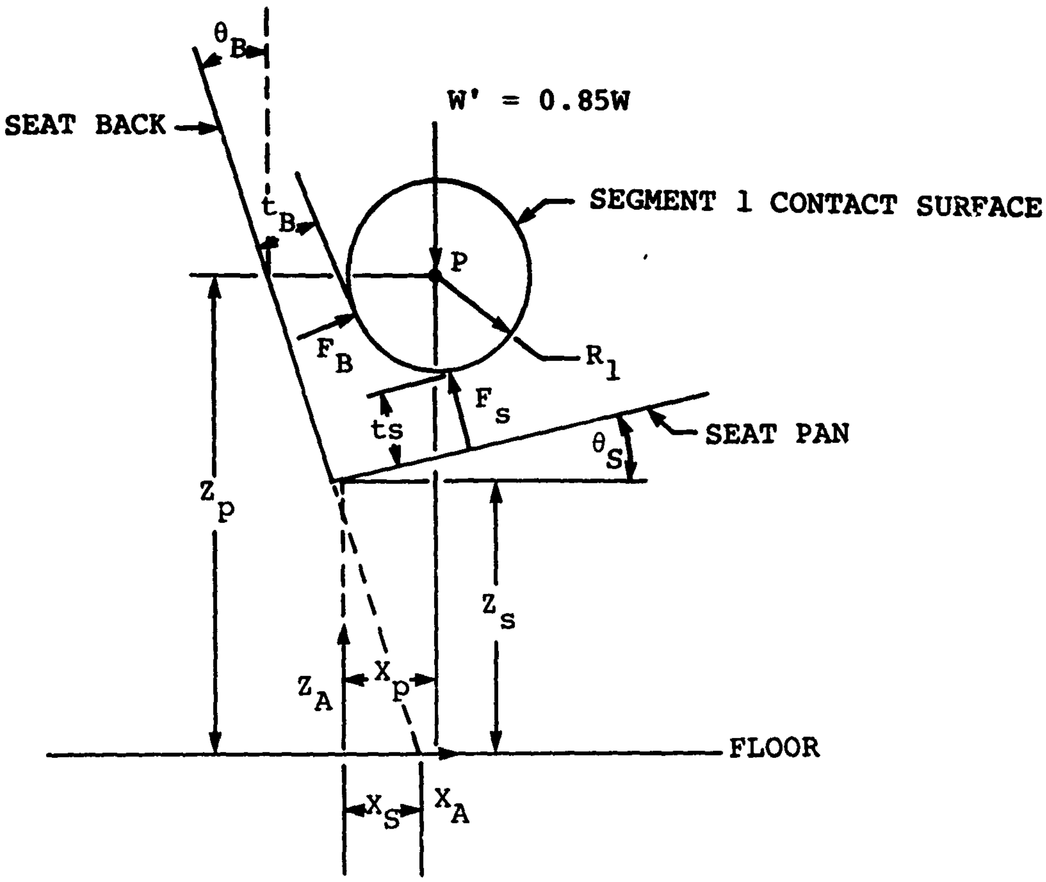


Figure 25. Forces Acting on Occupant Torso (Level Flight).

$$F_{X_A}: \quad F_B \cos \theta_B - F_S \sin \theta_S = 0$$

$$F_{Z_A}: \quad F_B \sin \theta_B + F_S \cos \theta_S = W' \qquad (82)$$

which can be solved for the cushion forces:

$$F_S = W' \cos \theta_S / \cos (\theta_B - \theta_S)$$

$$F_B = W' \sin \theta_S / \cos (\theta_B - \theta_S) \qquad (83)$$

Dimensional considerations permit the coordinates of point P to be written as functions of the thicknesses t_S and t_B of the compressed seat and back cushions, respectively.

$$Z_p = Z_S + (R_1 + t_S)/\cos\theta_S + X_p \tan\theta_S$$

$$X_p = X_S + (R_1 + t_B)/\cos\theta_B - Z_p \tan\theta_B \qquad (84)$$

which can be solved for X_p and Z_p to give

$$X_p = (f_1 \cos\theta_S - f_2 \sin\theta_B)/\cos(\theta_B - \theta_S)$$

$$Z_p = (f_1 \sin\theta_S + f_2 \cos\theta_B)/\cos(\theta_B - \theta_S)$$

where $f_1 = R_1 + t_B + X_S \cos\theta_B$

$$f_2 = R_1 + t_S + Z_S \cos\theta_S \qquad (85)$$

Since the force-deformation characteristics of the cushions are known from input data, the compressed thicknesses t_S and t_B can be calculated from equation (83). These values, when used in equation (85), give the coordinates of point P for the first approximation of level flight. The equilibrium (zero-load) lengths of the lap belt and shoulder belt(s) are calculated for the body in this position.

Next, the aircraft is rotated to the attitude specified by the input conditions of pitch, roll, and yaw. Nose-up pitch will tend to load the back cushion, and the analysis will be the same as that described above for level flight, except that the W' vector in figure 25 will have a component in the X_A direction.

Nose-down pitch, on the other hand, will tend to load the restraint system. An iterative procedure is used to determine the correct position for this case. Referring to figure 26, summing forces gives a set of transcendental equations

$$F_{X_A}: \quad W' \sin\theta_A - F_S \sin\theta_S - F_L \cos\theta_L = 0$$

$$F_{Z_A}: \quad -W' \cos\theta_A + F_S \cos\theta_S - F_L \sin\theta_L = 0 \qquad (86)$$

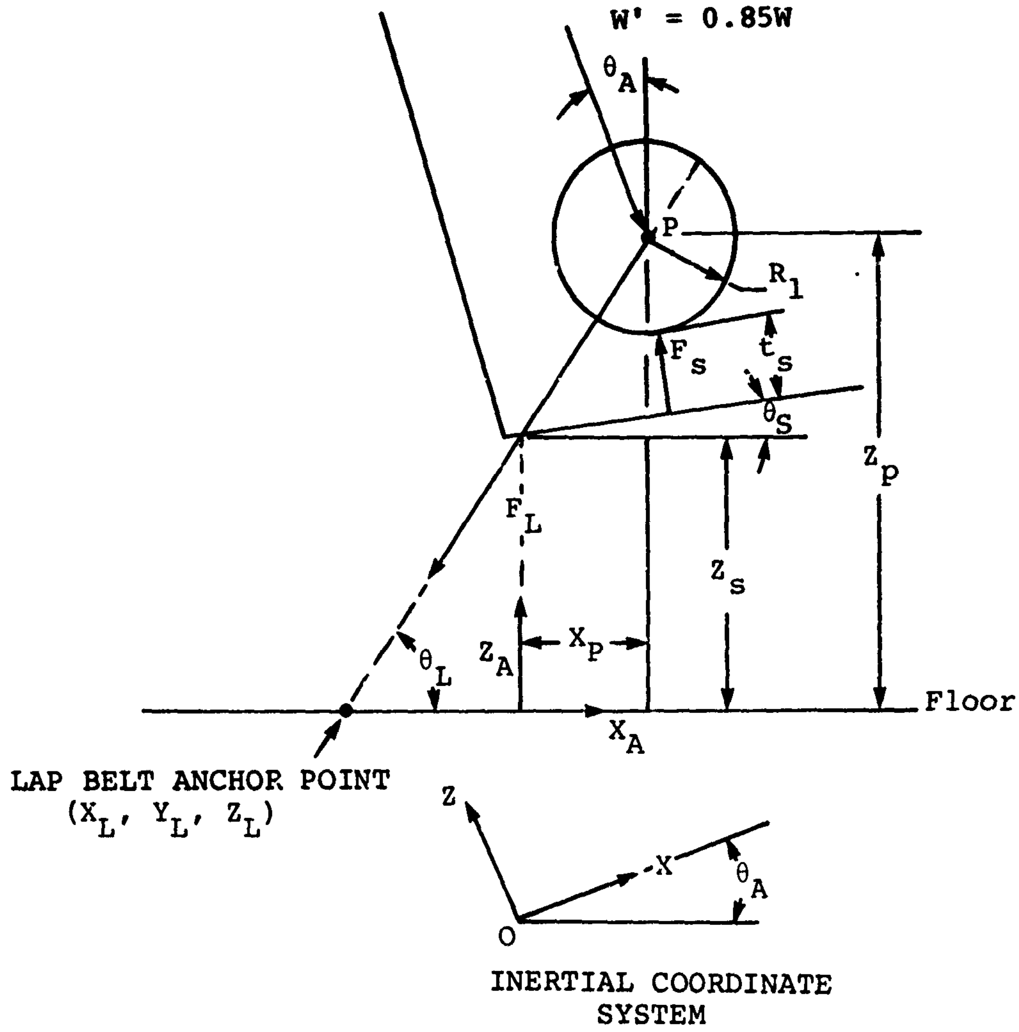


Figure 26. Forces Acting on Occupant Torso (Nose-Down Attitude).

where θ_L is the angle between the floor and the projection of the lap belt on the X_A - Z_A plane. An initial estimate to θ_L is made using the body position calculated by equation (85) for the level flight assumption. The angle is defined according to

$$\theta_L = \sin^{-1} \left[(Z_P - Z_L) / \sqrt{(X_P - X_L)^2 + (Z_P - Z_L)^2} \right] \qquad (87)$$

The forces in the seat cushion F_S and the lap belt F_L are determined using this value of θ_L in equation (86). From the input force-deformation characteristics for the seat cushion and lap belt, the deformations δ_S and δ_L are calculated. These deformations are used to determine new values of X_P and Z_P; this procedure amounts to permitting the body to further compress the seat cushion and slide forward into the lap belt. Following through the procedure, the new length for one side of the lap belt is

$$L_L = L_{Le} + \delta_L \tag{88}$$

where L_{Le} is the equilibrium length. The new value of X_P is given by

$$X_P = X_L + [(L_L - L_H)^2 - (Y_P - Y_L)^2]^{1/2} \cos \theta_L \tag{89}$$

where L_H is one-half the hip breadth and Y_P is the Y-coordinate of the right hip in the aircraft system. The new value of Z_P is computed for the new cushion thickness t_S using equation (84), which is repeated here for continuity:

$$Z_P = Z_S + (R_1 + t_S)/\cos \theta_S + X_P \tan \theta_S$$

The new occupant position, determined by equations (89) and (84) is used in equation (87) to recalculate the lap belt angle θ_L, and the procedure is repeated until two consecutive values of X_P agree by less than 5 percent. The coordinates of the mass center of segment 1 (X_1, Y_1, Z_1) are then calculated from X_P, Y_P, and Z_P.

At this point the generalized coordinates q_1 through q_{20} have been determined. The final task will be to determine the coordinates for the legs. Referring to figure 27, the angles γ_8 and θ_k can be found from simple geometric relationships among the dimensions shown. The Euler angles ψ_8, θ_8, and ϕ_8 are obtained from γ_8 by using equations (79) through (81), and the

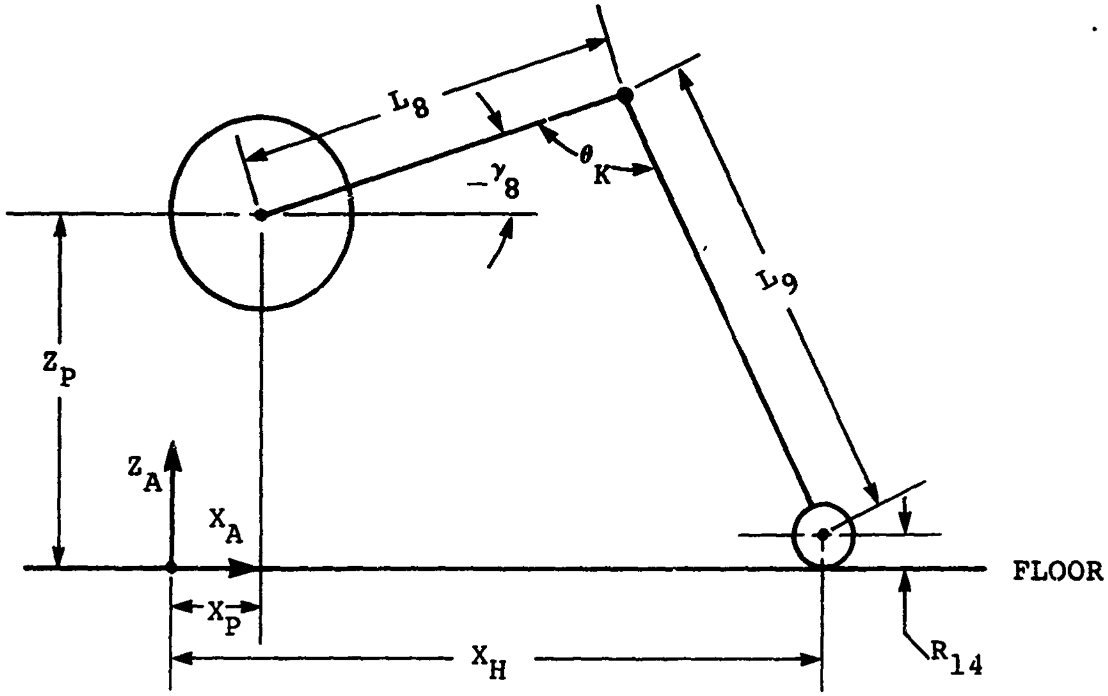


Figure 27. Leg Position.

corresponding coordinates for segment 10, by symmetry. The knee angles are given by

$$\alpha_m = \pi - \theta_k \qquad m = 9, 11 \tag{90}$$

to complete the initialization of the 28 generalized coordinates.

An alternative procedure, which is included in program SOM-LA to permit re-start from the final conditions of a previous run, bypasses the entire initialization procedure described here and uses input values of q_1 - q_{28}.

3.1.3 Restraint System Description

The restraint system used in the simulation may consist of a lap belt alone or combined with a single- or double-strap shoulder harness. The webbing force-elongation curve is approximated by three linear segments, which are described by input of four

points on the curve. The force is computed by linear interpolation in this table, as described in section 3.2.1. The slack in the webbing is also provided by input in units of length.

The anchor points for the lap belt and shoulder harness are located by input of rectangular coordinates in the aircraft reference system. For a double-strap shoulder harness the buckle, or point of connection to the lap belt, is assumed located on the mid-point of the lap belt. For a single shoulder belt, which may pass over either the left or the right shoulder, an input parameter locates the buckle by the length of webbing between the buckle and the lap belt anchor point. This length may be zero if the buckle attaches directly to a rigid anchor point.

3.1.4 Cushion Description

Input of a table of four forces and deflections describes the cushion characteristics. The equilibrium (zero load) thickness for both the seat and back cushions are also given.

3.1.5 Cockpit Description

For prediction of impact between the occupant and the cockpit interior, ten plane surfaces are used to describe the cockpit. As shown in figure 28, six of these surfaces are normal to the $X_A - Z_A$ plane and four are normal to the $Y_A - Z_A$ plane. The first five planes can be used to describe the environment of a crew seat, in which case they represent the firewall, instrument panel, and windscreen, or, for analysis of a passenger seat, they can be rearranged to describe a seat back. Input data include X and Z coordinates to define planes 1-6 and Y and Z coordinates for planes 7-10.

3.1.6 Crash Conditions

The aircraft crash conditions are defined by the initial velocity and attitude and the acceleration as a function of time. Six components of velocity are required: three translational in the aircraft coordinate system (V_{X_A}, V_{Y_A}, V_{Z_A}) and the yaw, pitch,

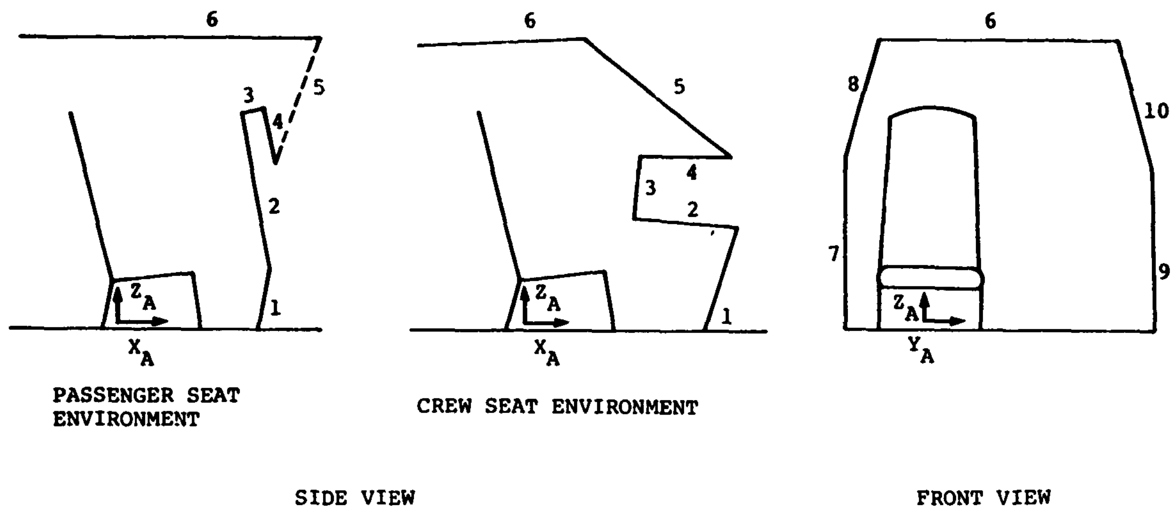


Figure 28. Planar Surface Approximation to Aircraft Interior.

and roll rates ($\dot{\psi}_A$, $\dot{\theta}_A$, $\dot{\phi}_A$). Each of the six acceleration components, which define the acceleration of the aircraft coordinate system, is described by sixteen points in time and acceleration. An example of an approximation to an actual acceleration pulse is illustrated in figure 29. Although many of the higher frequency oscillations observed in the actual pulse probably contribute little to the overall response of the occupant, the use of a large number of points reduces the effect of the investigator's subjectivity in the approximation.

3.1.7 Seat Design Information

The input data required to describe the seat consist of dimensions, material properties, and floor attachment strengths, as described earlier. For possible use in restraint system or cabin configuration analyses where seat response may be unimportant or seat design unknown, a rigid seat model can be selected. Input data for the rigid seat option consist only of locations of the seat pan and seat back.

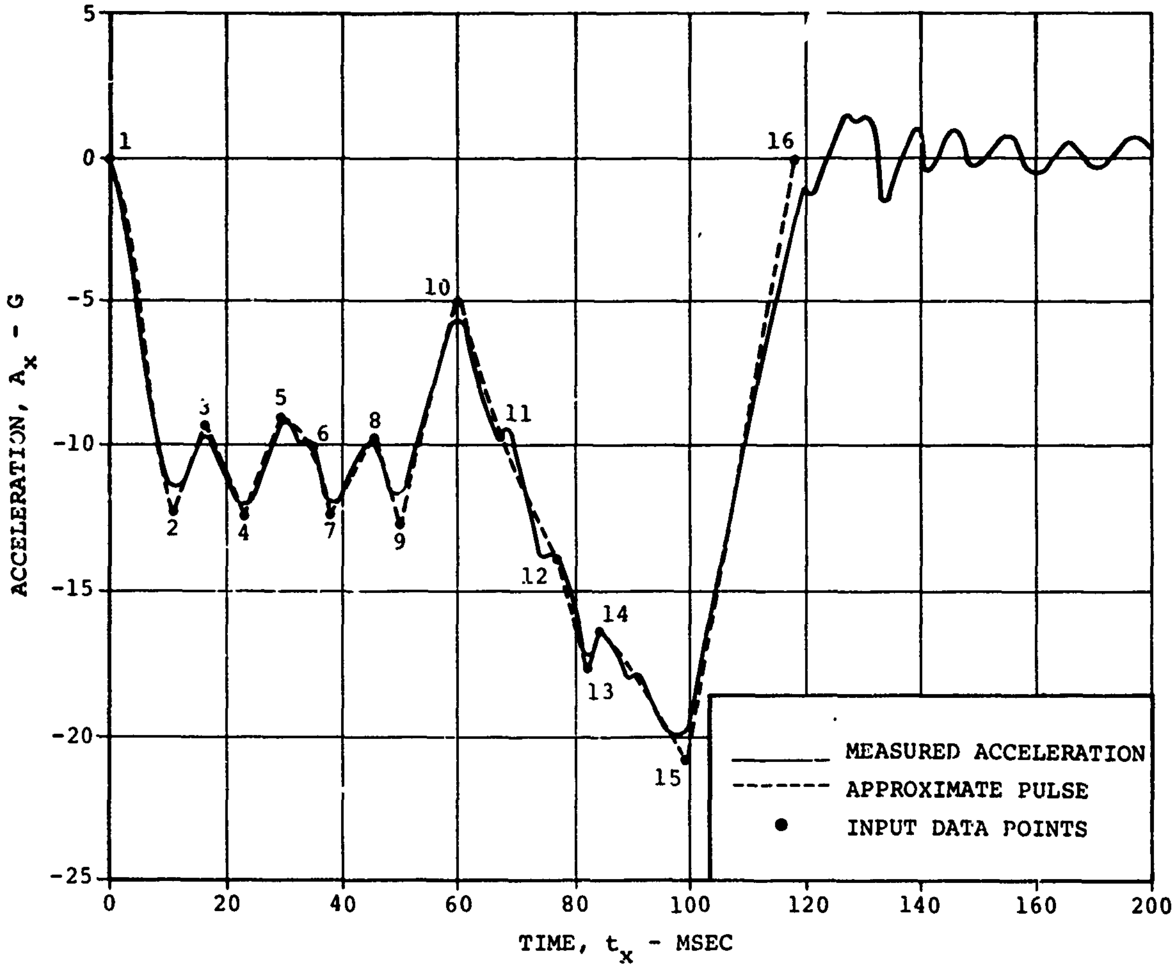


Figure 29. Approximation to Acceleration Pulse.

3.2 PROGRAM SOLUTION PROCEDURE

The first operation in each solution step includes the calculation of new values for the aircraft acceleration components and their subsequent integration to obtain aircraft velocity and displacement components. Then the matrix form of the equations of motion, using equation (19)

$$[A(q)]\ \{\ddot{q}\} = \{B(\dot{q},q)\} + \{P(q)\} + \{R(\dot{q},q)\} + \{Q(\dot{q},q)\}$$

are set up for solution and solved, as discussed below.

3.2.1 Setup of Equations of Motion

The elements of [A], {B}, {P}, and {R} are calculated from the expressions presented in appendixes D through F, using the current values of the generalized coordinates and velocities. The elements of {Q}, which is the vector of generalized external forces, are calculated, as discussed in section 2.1.3. The external forces depend on displacements of the aircraft, which determine the motion of the seat, floor, and restraint system anchor points relative to the body. From these displacements new deflections of the cushions, floor, and restraint system are calculated. The forces are then calculated by linear interpolation in a table of forces and deformations are provided as input data.

The model used for calculating all forces is illustrated in figure 30, which shows the force-elongation characteristics for a typical restraint system webbing. The experimental curve is approximated by three linear segments. Deformation between points 1 and 2 is considered elastic, so that unloading would proceed down the loading curve. However, beyond point 2 a decrease in deformation will cause the member to unload along a fourth-order curve between the point where the unloading starts and the origin. If the deformation should return to zero, reloading takes place along the original loading curve, i.e., from 1 to 2, etc. However, if the deformation increases again prior to reaching zero the member will reload along a straight line from the point where reloading starts to the point from which unloading had started.

3.2.2 Solution of Equations of Motion

The system of 28 equations is solved for the generalized accelerations by first combining the vectors on the right-hand side:

$$[A]\ \{q\} = \{B'\} \tag{91}$$

where $\{B'\} = \{B\} + \{P\} + \{R\} + \{Q\}$

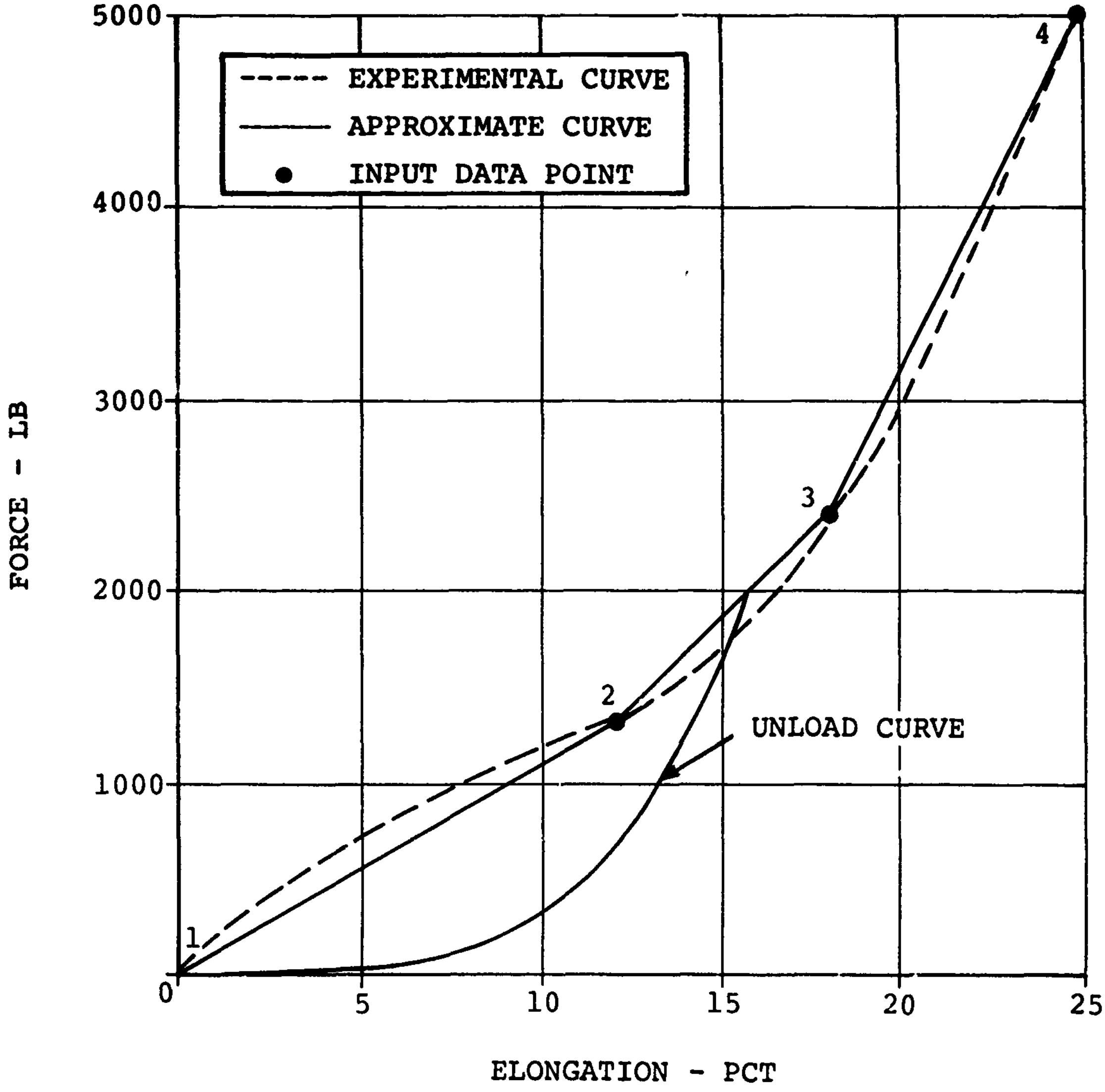


Figure 30. Example of Webbing Force-Elongation Curve.

and using Gaussian elimination with positioning for size:

$$\{\ddot{q}\} = [A]^{-1} \{B\} \tag{92}$$

The resulting set of 28 second-order differential equations have the general form

$$\begin{gathered} \ddot{q}_j = f_j \, (t, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{28}, q_1, q_2, \ldots, q_{28}) \\ q_j \, (t=0) = q_{j0} \; \dot{q}_j \, (t=0) = \dot{q}_{j0} \qquad j = 1, 2, \ldots, 28 \end{gathered} \tag{93}$$

These equations can be rewritten as 56 first-order equations having the general form

$$\dot{y}_j = f_j\ (t, y_1, y_2, \ldots, y_{28}, q_1, q_2, \ldots, q_{28})$$

$$\dot{q}_j = y_j$$

$$y_j\ (t=0) = \dot{q}_{j0} \qquad q_j\ (t=0) = q_{j0} \tag{94}$$

Numerical integration of this set of equations is accomplished, using the Adams-Moulton predictor-corrector method with a variable step size. This method uses the difference equations

$$y^{(p)}_{j,n+1} = y_{j,n} + \frac{h}{24}\,(55f_{j,n} - 59f_{j,n-1} + 37f_{j,n-2} - 9f_{j,n-3}) \tag{95}$$

as the predictor and

$$y^{(c)}_{j,n+1} = y_{j,n} + \frac{h}{24}\,(9f^{(p)}_{j,n+1} + 19f_{j,n} - 5f_{j,n-1} + f_{j,n-2}) \tag{96}$$

as the corrector. Starting values are provided by the classical fourth-order Runge-Kutta method. Input data includes upper and lower error bounds for the solution. Error bounds for each variable are calculated and compared at each step with the difference between the predicted value $y_j^{(p)}$ and the corrected value $y_j^{(c)}$. If this difference exceeds the upper bound for any j, the step size is halved. If this difference is less than the minimum error bound for all j and for three successive steps, the step size is doubled. Halving the step size is accomplished by interpolation of past data, whereas, doubling is effected by alternate selection of past data. The solution can be run with a fixed step size by making the upper and lower error bounds prohibitively large and small, respectively, or by using equal values for the maximum and minimum step size, which are also included among input data.

3.3 PROGRAM OUTPUT

Output data consist of eight blocks of information that are selected for printing by user input. The data include time histories of the variables, which are simply stored during the solution at pre-determined print intervals as follows:

1. Occupant segment positions
2. Occupant segment velocities
3. Occupant segment accelerations
4. Restraint system loads (tensile loads in webbing and resultant normal loads on pelvis and chest)
5. Seat deflections at critical points
6. Floor reactions

and additional information as follows:

7. Details of contact between the occupant and the aircraft interior
8. Injury criteria

The last two blocks of output data will be discussed in further detail.

3.3.1 Impact Prediction

For prediction of impact between the occupant and the cockpit interior, 23 surfaces are defined on the body. These surfaces were illustrated in figure 15, and their dimensions discussed in section 2.1.4.4.

The distance between each of these occupant contact surfaces and the aircraft cockpit surfaces is calculated, during execution of the program. When contact occurs between an occupant surface and a contact surface, the time and relative velocity of impact are computed and stored for printing. The impact conditions determined in this way can be used in evaluation of injury potential for a given cockpit configuration.

3.3.2 Injury Criteria

The injury criteria used in the program were selected as the most suitable for aircraft crash analysis. The dynamic response index (DRI) is used to determine the probability of spinal injury due to a vertical acceleration parallel to the spine (reference 10).

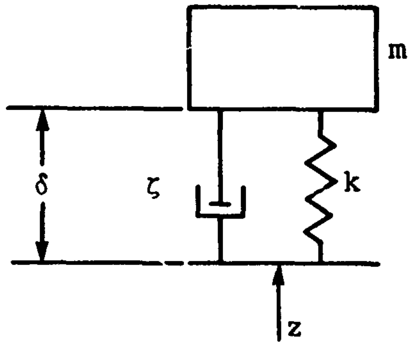
In this model, the response of the body to acceleration parallel to the spine is modeled by a single lumped-mass, damped-spring system as shown in figure 31, or, in other words, the total body mass that acts on the vertebral column to cause deformation is represented by the single mass. In general, the motion of the system shown in figure 31 obeys the relationship

$$\frac{d^2\delta}{dt} + 2\zeta\omega_n \frac{d\delta}{dt} + \omega_n^2 \delta = Z \tag{97}$$

The solution, the deflection δ, is representative of the deformation of the spine, and the last term on the left-hand side of equation (97), divided by the gravitational acceleration, is the DRI. The properties used in the model were derived from tests involving human subjects and cadavers. For example, the spring stiffness k was determined from tests of human cadaver vertebral segments; damping ratios were determined from measurements of mechanical impedance of human subjects during vibration and impact.

In the occupant model used in program SOM-LA, it is assumed that the spring in figure 31 represents the lumbar spine. This is a reasonable assumption, since compression fractures that occur in vertical impact often involve this part of the spinal column. Therefore, the mass m in figure 31 is the sum of segments 2, 3, 4, 5, 6, and 7. The input acceleration is the component of the acceleration of segment 1 in the z_1-direction:

$$\ddot{z}_1 = \ddot{X}_1 T_{13}^1 + \ddot{Y}_1 T_{23}^1 + \ddot{Z}_1 T_{33}^1 \tag{98}$$



*Dynamic Response Index

m = mass (lb-sec^2/in.)
δ = deflection (in.)
ζ = damping ratio
k = stiffness (lb/in.)
z = acceleration input (in./sec^2)

$$*DRI = \frac{\omega_n^2 \, \delta_{max}}{g}$$

ω_n = natural frequency of the analog = $\sqrt{k/m}$ (rad/sec)

g = 386 in./sec^2

Figure 31. Model Used for Prediction of Spinal Injury (From Reference 10).

so that the equation to be solved is

$$\ddot{\delta} + 2\zeta\omega_n\dot{\delta} + \omega_n^2\delta = \ddot{z} \qquad (99)$$

where $\zeta = 0.224$

$\omega_n = 52.9$ rad/sec.

The DRI is calculated at each step by

$$DRI = \omega_n^2\delta/g \qquad (100)$$

and the maximum value is stored for output.

Another injury criterion is the Severity Index (SI) (reference 11), which is calculated for the head and chest according to

$$SI = \int_{t_o}^{t_f} a^{2.5} \, dt \qquad (101)$$

where a = acceleration as a function of time

n = weighting factor ($n>1$)

t = time

The severity index has been validated for frontal impacts of the head-face with $n=2.5$. Although such a validation has not been performed for other parts of the body, the SI is calculated for the chest as it may serve a useful function in determining relative levels of injury potential due to an acceleration environment.

Finally, the Head Injury Criterion (HIC) contained in Federal Motor Vehicle Safety Standard 208 is calculated according to

$$\mathrm{HIC} = \max \left(\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} a\,dt \right)^{2.5} (t_2 - t_1) \tag{102}$$

where a is the resultant head acceleration and t_1 and t_2 are any two points in time during the crash event.

4.0 SUMMARY

A three-dimensional mathematical model of an aircraft seat, occupant, and restraint system has been developed as an aid to the development of crashworthy seats and restraint systems for general aviation aircraft. The occupant model consists of eleven rigid mass segments whose dimensions and inertial properties have been determined from studies of human body anthropometry and kinematics. The seat model is made up of beam and membrane elements with provision for simulating plastic behavior by the introduction of plastic hinges in the beams.

A user-oriented computer program called Seat Occupant Model-Light Aircraft (SOM-LA) based on the three-dimensional model has been developed for use by engineers concerned with design and analysis of general aviation seats and restraint systems. Detailed descriptions of both are used as input data. The response of the seat and occupant, restraint system loads, and various injury criteria are predicted for any given set of crash conditions.

Results of the computer program validation will be covered in a report entitled "Validation of the SOM-LA Computer Program."

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APPENDIX A

OCCUPANT SEGMENT POSITION AND VELOCITY COMPONENTS

A.1 SEGMENT POSITION COMPONENTS

Referring to figure 13, which has been repeated here as figure A-1 for convenience, and using $\bar{\rho}_n = L_n - \bar{\rho}_n$ the absolute position of the mass center of each body segment is given below. The elements of the transformation matrices $[T^n]$ are functions of the generalized coordinates, as given by equations (6), (9), and (10).

Segment 1:

(X_1, Y_1, Z_1), the coordinates of the reference point on the body are the generalized coordinates (q_1, q_2, q_3).

Segment 2:

$$\begin{Bmatrix} X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} \\ T^1 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_1 \end{Bmatrix} + \begin{bmatrix} \\ T^2 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_2 \end{Bmatrix}$$

Segment 3:

$$\begin{Bmatrix} X_3 \\ Y_3 \\ Z_3 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} \\ T^1 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_1 \end{Bmatrix} + \begin{bmatrix} \\ T^2 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L_2 \end{Bmatrix} + \begin{bmatrix} \\ T^3 \\ \\ \end{bmatrix} \begin{Bmatrix} L_c \\ 0 \\ \rho_3 \end{Bmatrix}$$

Segment 4:

$$\begin{Bmatrix} X_4 \\ Y_4 \\ Z_4 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} \\ T^1 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_1 \end{Bmatrix} + \begin{bmatrix} \\ T^2 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ -L_s \\ L_2 \end{Bmatrix} + \begin{bmatrix} \\ T^4 \\ \\ \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\rho_4 \end{Bmatrix}$$

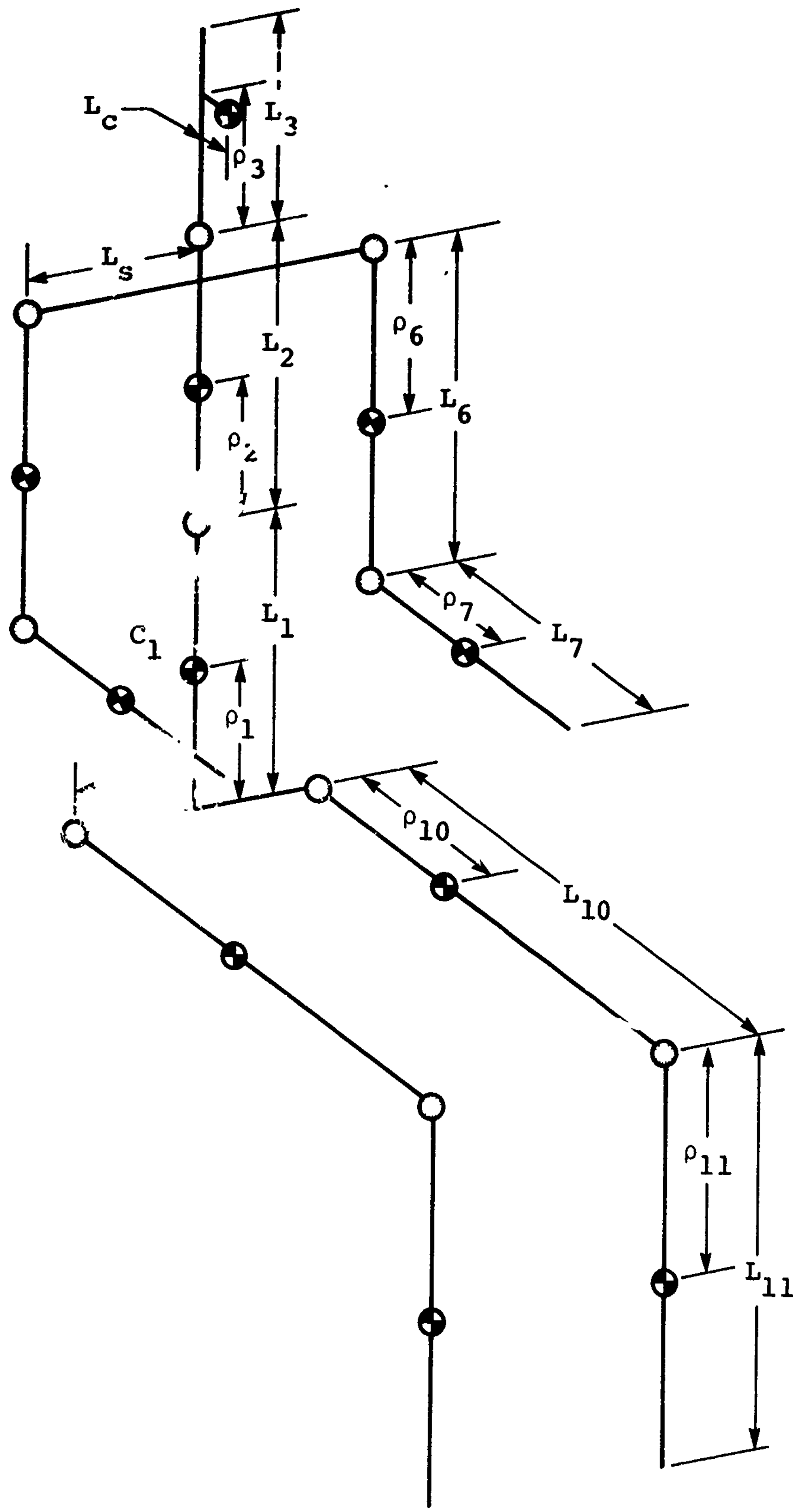


Figure A-1. Body Segment Lengths and Mass Center Locations.

Segment 5:

$$\begin{Bmatrix} X_5 \\ Y_5 \\ Z_5 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_1 \end{Bmatrix} + \begin{bmatrix} T^2 \end{bmatrix} \begin{Bmatrix} 0 \\ -L_s \\ L_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} T^4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -L_4 \end{Bmatrix} + \begin{bmatrix} T^5 \end{bmatrix} \begin{Bmatrix} \rho_5 \\ 0 \\ 0 \end{Bmatrix}$$

Segment 6:

$$\begin{Bmatrix} X_6 \\ Y_6 \\ Z_6 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_1 \end{Bmatrix} + \begin{bmatrix} T^2 \end{bmatrix} \begin{Bmatrix} 0 \\ +L_s \\ L_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} T^6 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\rho_6 \end{Bmatrix}$$

Segment 7:

$$\begin{Bmatrix} X_7 \\ Y_7 \\ Z_7 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \bar{\rho}_1 \end{Bmatrix} + \begin{bmatrix} T^2 \end{bmatrix} \begin{Bmatrix} 0 \\ L_s \\ L_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} T^6 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -L_6 \end{Bmatrix} + \begin{bmatrix} T^7 \end{bmatrix} \begin{Bmatrix} \rho_7 \\ 0 \\ 0 \end{Bmatrix}$$

Segment 8:

$$\begin{Bmatrix} X_8 \\ Y_8 \\ Z_8 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ -L_H \\ -\rho_1 \end{Bmatrix} + \begin{bmatrix} T^8 \end{bmatrix} \begin{Bmatrix} \rho_8 \\ 0 \\ 0 \end{Bmatrix}$$

Segment 9:

$$\begin{Bmatrix} X_9 \\ Y_9 \\ Z_9 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ -L_H \\ -\rho_1 \end{Bmatrix} + \begin{bmatrix} T^8 \end{bmatrix} \begin{Bmatrix} L_8 \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} T^9 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\rho_9 \end{Bmatrix}$$

Segment 10:

$$\begin{Bmatrix} X_{10} \\ Y_{10} \\ Z_{10} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ L_H \\ -\rho_1 \end{Bmatrix} + \begin{bmatrix} T^{10} \end{bmatrix} \begin{Bmatrix} \rho_{10} \\ 0 \\ 0 \end{Bmatrix}$$

Segment 11:

$$\begin{Bmatrix} X_{11} \\ Y_{11} \\ Z_{11} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \begin{bmatrix} T^1 \end{bmatrix} \begin{Bmatrix} 0 \\ L_H \\ -\rho_1 \end{Bmatrix} + \begin{bmatrix} T^{10} \end{bmatrix} \begin{Bmatrix} L_{10} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} T^{11} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\rho_{11} \end{Bmatrix}$$

A.2 SEGMENT VELOCITIES

The velocity components for the mass center of each segment are given below. Here, the symmetry of the body has been used for convenience in combining similar terms at a later point. By symmetry,

$$L_6 = L_4 \qquad \rho_6 = \rho_4$$

$$L_7 = L_5 \qquad \rho_7 = \rho_5$$

$$L_{10} = L_8 \qquad \rho_{10} = \rho_8$$

$$L_{11} = L_9 \qquad \rho_{11} = \rho_9$$

$$\dot{X}_1 = \dot{X}_1$$

$$\dot{Y}_1 = \dot{Y}_1$$

$$\dot{Z}_1 = \dot{Z}_1$$

$$\dot{X}_2 = \dot{X}_1 + \bar{\rho}_1\dot{T}_{13}{}^1 + \rho_2\dot{T}_{13}{}^2$$

$$\dot{Y}_2 = \dot{Y}_1 + \bar{\rho}_1\dot{T}_{23}{}^1 + \rho_2\dot{T}_{23}{}^2$$

$$\dot{Z}_2 = \dot{Z}_1 + \bar{\rho}_1\dot{T}_{33}{}^1 + \rho_2\dot{T}_{33}{}^2$$

$$\dot{X}_3 = \dot{X}_1 + \bar{\rho}_1\dot{T}_{13}{}^1 + L_2\dot{T}_{13}{}^2 + \rho_3\dot{T}_{13}{}^3 + L_c\dot{T}_{11}{}^3$$

$$\dot{Y}_3 = \dot{Y}_1 + \bar{\rho}_1\dot{T}_{23}{}^1 + L_2\dot{T}_{23}{}^2 + \rho_3\dot{T}_{23}{}^3 + L_c\dot{T}_{21}{}^3$$

$$\dot{Z}_3 = \dot{Z}_1 + \bar{\rho}_1\dot{T}_{33}{}^1 + L_2\dot{T}_{33}{}^2 + \rho_3\dot{T}_{33}{}^3 + L_c\dot{T}_{31}{}^3$$

$$\dot{X}_4 = \dot{X}_1 + \bar{\rho}_1\dot{T}_{13}{}^1 + L_2\dot{T}_{13}{}^2 - L_s\dot{T}_{12}{}^2 - \rho_4\dot{T}_{13}{}^4$$

$$\dot{Y}_4 = \dot{Y}_1 + \bar{\rho}_1\dot{T}_{23}{}^1 + L_2\dot{T}_{23}{}^2 - L_s\dot{T}_{22}{}^2 - \rho_4\dot{T}_{23}{}^4$$

$$\dot{Z}_4 = \dot{Z}_1 + \bar{\rho}_1\dot{T}_{33}{}^1 + L_2\dot{T}_{33}{}^2 - L_s\dot{T}_{32}{}^2 - \rho_4\dot{T}_{33}{}^4$$

$$\dot{X}_5 = \dot{X}_1 + \bar{\rho}_1\dot{T}_{13}{}^1 + L_2\dot{T}_{13}{}^2 - L_s\dot{T}_{12}{}^2 - L_4\dot{T}_{13}{}^4 + \rho_5\dot{T}_{11}{}^5$$

$$\dot{Y}_5 = \dot{Y}_1 + \bar{\rho}_1\dot{T}_{23}^{1} + L_2\dot{T}_{23}^{2} - L_s\dot{T}_{22}^{2} - L_4\dot{T}_{23}^{4} + \rho_5\dot{T}_{21}^{5}$$

$$\dot{Z}_5 = \dot{Z}_1 + \rho_1\dot{T}_{33}^{1} + L_2\dot{T}_{33}^{2} - L_s\dot{T}_{23}^{2} - L_4\dot{T}_{33}^{4} + \rho_5\dot{T}_{31}^{5}$$

$$\dot{X}_6 = \dot{X}_1 + \bar{\rho}_1\dot{T}_{13}^{1} + L_2\dot{T}_{13}^{2} + L_s\dot{T}_{12}^{2} - \rho_4\dot{T}_{13}^{6}$$

$$\dot{Y}_6 = \dot{Y}_1 + \bar{\rho}_1\dot{T}_{23}^{1} + L_2\dot{T}_{23}^{2} + L_s\dot{T}_{22}^{2} - \rho_4\dot{T}_{23}^{6}$$

$$\dot{Z}_6 = \dot{Z}_1 + \bar{\rho}_1\dot{T}_{33}^{1} + L_2\dot{T}_{23}^{2} + L_s\dot{T}_{32}^{2} - \rho_4\dot{T}_{33}^{6}$$

$$\dot{X}_7 = \dot{X}_1 + \bar{\rho}_1\dot{T}_{13}^{1} + L_2\dot{T}_{13}^{2} + L_s\dot{T}_{12}^{2} - L_4\dot{T}_{13}^{6} + \rho_5\dot{T}_{11}^{7}$$

$$\dot{Y}_7 = \dot{Y}_1 + \bar{\rho}_1\dot{T}_{23}^{1} + L_2\dot{T}_{23}^{2} + L_s\dot{T}_{22}^{2} - L_4\dot{T}_{23}^{6} + \rho_5\dot{T}_{21}^{7}$$

$$\dot{Z}_7 = \dot{Z}_1 + \rho_1\dot{T}_{33}^{1} + L_2\dot{T}_{33}^{2} + L_s\dot{T}_{32}^{2} - L_4\dot{T}_{33}^{6} + \rho_5\dot{T}_{31}^{7}$$

$$\dot{X}_8 = \dot{X}_1 - \rho_1\dot{T}_{13}^{1} - L_H\dot{T}_{12}^{1} + \rho_8\dot{T}_{11}^{8}$$

$$\dot{Y}_8 = \dot{Y}_1 - \rho_1\dot{T}_{23}^{1} - L_H\dot{T}_{22}^{1} + \rho_8\dot{T}_{21}^{8}$$

$$\dot{Z}_8 = \dot{Z}_1 - \rho_1\dot{T}_{33}^{1} - L_H\dot{T}_{32}^{1} + \rho_8\dot{T}_{31}^{8}$$

$$\dot{X}_9 = \dot{X}_1 - \rho_1\dot{T}_{13}^{1} - L_H\dot{T}_{12}^{1} + L_8\dot{T}_{11}^{8} - \rho_9\dot{T}_{13}^{9}$$

$$\dot{Y}_9 = \dot{Y}_1 - \rho_1\dot{T}_{23}^{1} - L_H\dot{T}_{22}^{1} + L_8\dot{T}_{21}^{8} - \rho_9\dot{T}_{23}^{9}$$

$$\dot{Z}_9 = \dot{Z}_1 - \rho_1\dot{T}_{33}{}^1 - L_H\dot{T}_{32}{}^1 + L_8\dot{T}_{31}{}^8 - \rho_9\dot{T}_{33}{}^9$$

$$\dot{X}_{10} = \dot{X}_1 - \rho_1\dot{T}_{13}{}^1 + L_H\dot{T}_{12}{}^1 + \rho_8\dot{T}_{11}{}^{10}$$

$$\dot{Y}_{10} = \dot{Y}_1 - \rho_1\dot{T}_{23}{}^1 + L_H\dot{T}_{22}{}^1 + \rho_8\dot{T}_{21}{}^{10}$$

$$\dot{Z}_{10} = \dot{Z}_1 - \rho_1\dot{T}_{33}{}^1 + L_H\dot{T}_{32}{}^1 + \rho_8\dot{T}_{31}{}^{10}$$

$$\dot{X}_{11} = \dot{X}_1 - \rho_1\dot{T}_{13}{}^1 + L_H\dot{T}_{12}{}^1 + L_8\dot{T}_{11}{}^{10} - \rho_9\dot{T}_{13}{}^{11}$$

$$\dot{Y}_{11} = \dot{Y}_1 - \rho_1\dot{T}_{23}{}^1 + L_H\dot{T}_{22}{}^1 + L_8\dot{T}_{21}{}^{10} - \rho_9\dot{T}_{23}{}^{11}$$

$$\dot{Z}_{11} = \dot{Z}_1 - \rho_1\dot{T}_{33}{}^1 + L_H\dot{T}_{32}{}^1 + L_8\dot{T}_{31}{}^{10} - \rho_9\dot{T}_{33}{}^{11}$$

APPENDIX B

OCCUPANT KINETIC ENERGY

Occupant Kinetic Energy

The system kinetic energy T is written below as a function of the generalized coordinates. Prior to the kinetic energy a set of 50 constants, functions of body properties, are defined.

$$C_1 = \sum_{j=1}^{11} M_j$$

$$C_2 = \bar{\rho}_1^2 \sum_{j=2}^{7} M_j + \rho_1^2 \sum_{j=8}^{11} M_j$$

$$C_3 = \rho_2^2 M_2 + L_2^2 \sum_{j=3}^{7} M_j$$

$$C_4 = L_c^2 M_3$$

$$C_5 = L_s^2 \sum_{j=4}^{7} M_j$$

$$C_6 = L_H^2 \sum_{j=8}^{11} M_j$$

$$C_7 = M_4 \rho_4^2 + M_5 L_4^2$$

$$C_8 = M_5 \rho_5^2$$

$$C_9 = M_8 \rho_8^2 + M_9 L_8^2$$

$$C_{10} = M_9 \rho_9^2$$

$$C_{11} = \bar{\rho}_1 \sum_{j=2}^{7} M_j - \rho_1 \sum_{j=8}^{11} M_j$$

$$C_{12} = \rho_2 M_2 + L_2 \sum_{j=3}^{7} M_j$$

$$C_{13} = \rho_3 M_3$$

$$C_{14} = L_c M_3$$

$$C_{15} = M_4 \rho_4 + M_5 L_4$$

$$C_{16} = M_5 \rho_5$$

$$C_{17} = M_8 \rho_8 + M_9 L_8$$

$$C_{18} = M_9 \rho_9$$

$$C_{19} = (\rho_2 M_2 + L_2 \sum_{j=3}^{7} M_j)\ \bar{\rho}_1$$

$$C_{20} = M_3 \bar{\rho}_1 \rho_3$$

$$C_{21} = M_3 \bar{\rho}_1 L_c$$

$$C_{22} = M_3 L_2 \rho_3$$

$$C_{23} = M_3 L_2 L_c = C_{14} L_2$$

$$C_{24} = M_3 \rho_3 L_c = C_{14} \rho_3$$

$$C_{25} = (M_4 \rho_4 + M_5 L_4)\ \bar{\rho}_1 = C_{15} \bar{\rho}_1$$

$$C_{26} = (M_4 \rho_4 + M_5 L_4)\ L_2 = C_{15} L_2$$

$$C_{27} = (M_4\rho_4 + M_5L_4)\ L_s = C_{15}L_s$$

$$C_{28} = M_5\bar{\rho}_1\rho_5 = C_{16}\bar{\rho}_1$$

$$C_{29} = M_5L_2\rho_5 = C_{16}L_2$$

$$C_{30} = M_5L_s\rho_5 = C_{16}L_s$$

$$C_{31} = M_5L_4\rho_5 = C_{16}L_4$$

$$C_{32} = (M_8\rho_8 + M_9L_8)\ \rho_1 = C_{17}\rho_1$$

$$C_{33} = (M_8\rho_8 + M_9L_8)\ L_H = C_{17}L_H$$

$$C_{34} = M_9\rho_1\rho_9 = C_{18}\rho_1$$

$$C_{35} = M_9L_H\rho_9 = C_{18}L_H$$

$$C_{36} = M_9L_8\rho_9 = C_{18}L_8$$

$$C_{37} = I_{x_1} = I_{z_1}$$

$$C_{38} = I_{y_1}$$

$$C_{39} = I_{x_2}$$

$$C_{40} = I_{y_2} = I_{z_2}$$

$$C_{41} = I_{x_3} = I_{y_3}$$

$$C_{42} = I_{z_3}$$

$$C_{43} = I_{x_4} = I_{y_4} = I_{x_6} = I_{y_6}$$

$$C_{44} = I_{z_4} = I_{z_6}$$

$$C_{45} = I_{x_5} = I_{x_7}$$

$$C_{46} = I_{y_5} = I_{z_5} = I_{y_7} = I_{z_7}$$

$$C_{47} = I_{x_8} = I_{x_{10}}$$

$$C_{48} = I_{y_8} = I_{z_8} = I_{y_{10}} = I_{z_{10}}$$

$$C_{49} = I_{x_9} = I_{y_9} = I_{x_{11}} = I_{y_{11}}$$

$$C_{50} = I_{z_9} = I_{z_{11}}$$

Kinetic Energy:

$$T = 1/2\ C_1\ \left[(\dot{X}_1)^2 + (\dot{Y}_1)^2 + (\dot{Z}_1)^2\right]$$

$$+ 1/2\ C_2\ \left[(\dot{T}_{13}{}^2)^2 + (\dot{T}_{23}{}^2)^2 + (\dot{T}_{33}{}^1)^2\right]$$

$$+ 1/2\ C_3\ \left[(\dot{T}_{13}{}^2)^2 + (\dot{T}_{23}{}^2)^2 + (\dot{T}_{33}{}^1)^2\right]$$

$$+ 1/2\ C_{13}\rho_3\ \left[(\dot{T}_{13}{}^3)^2 + (\dot{T}_{23}{}^3)^2 + (\dot{T}_{33}{}^3)^2\right]$$

$$+ 1/2\ C_4\ \left[(\dot{T}_{11}{}^3)^2 + (\dot{T}_{21}{}^3)^2 + (\dot{T}_{31}{}^3)^2\right]$$

$$+ 1/2\ C_5\ \left[(\dot{T}_{12}{}^2)^2 + (\dot{T}_{22}{}^2)^2 + (\dot{T}_{32}{}^2)^2\right]$$

$$+ 1/2\ C_6\ \left[(\dot{T}_{12}{}^1)^2 + (\dot{T}_{22}{}^1)^2 + (\dot{T}_{32}{}^1)^2\right]$$

$$+ 1/2\ C_7\ \left[(\dot{T}_{13}{}^4)^2 + (\dot{T}_{23}{}^4)^2 + (\dot{T}_{33}{}^4)^2\right.$$

$$\left. + (\dot{T}_{13}{}^6)^2 + (\dot{T}_{23}{}^6)^2 + (\dot{T}_{33}{}^6)^2\right]$$

$$+ 1/2\ C_8\ \left[(\dot{T}_{11}{}^{45})^2 + (\dot{T}_{21}{}^{45})^2 + (\dot{T}_{31}{}^{45})^2\right.$$

$$\left. + (\dot{T}_{11}{}^{67})^2 + (\dot{T}_{21}{}^{67})^2 + (\dot{T}_{31}{}^{67})^2\right]$$

$$+ 1/2\ C_9\ \left[(\dot{T}_{11}{}^8)^2 + (\dot{T}_{21}{}^8)^2 + (\dot{T}_{31}{}^8)^2\right.$$

$$\left. + (\dot{T}_{11}{}^{10})^2 + (\dot{T}_{21}{}^{10})^2 + (\dot{T}_{31}{}^{10})^2\right]$$

$$+ 1/2\ C_{10}\ \left[(\dot{T}_{13}{}^{89})^2 + (\dot{T}_{23}{}^{89})^2 + (\dot{T}_{33}{}^{89})^2\right.$$

$$\left. + (\dot{T}_{13}{}^{1011})^2 + (\dot{T}_{23}{}^{1011})^2 + (\dot{T}_{33}{}^{1011})^2\right]$$

$$+ C_{11}\ (\dot{X}_1\dot{T}_{13}{}^{1} + \dot{Y}_1\dot{T}_{23}{}^{1} + \dot{Z}_1\dot{T}_{33}{}^{1})$$

$$+ C_{12}\ (\dot{X}_1\dot{T}_{13}{}^{2} + \dot{Y}_1\dot{T}_{23}{}^{2} + \dot{Z}_1\dot{T}_{33}{}^{2})$$

$$+ C_{13}\ (\dot{X}_1\dot{T}_{13}{}^{2} + \dot{Y}_1\dot{T}_{23}{}^{3} + \dot{Z}_1\dot{T}_{33}{}^{3})$$

$$+ C_{14}\ (\dot{X}_1\dot{T}_{11}{}^{3} + \dot{Y}_1\dot{T}_{21}{}^{3} + \dot{Z}_1\dot{T}_{31}{}^{3})$$

$$- C_{15}\ (\dot{X}_1\dot{T}_{13}{}^{4} + \dot{Y}_1\dot{T}_{23}{}^{4} + \dot{Z}_1\dot{T}_{33}{}^{4})$$

$$+ \dot{X}_1\dot{T}_{13}{}^{6} + \dot{Y}_1\dot{T}_{23}{}^{6} + \dot{Z}_1\dot{T}_{33}{}^{6})$$

$$+ C_{16}\ (\dot{X}_1\dot{T}_{11}{}^{45} + \dot{Y}_1\dot{T}_{21}{}^{45} + \dot{Z}_1\dot{T}_{31}{}^{45}$$

$$+ \dot{X}_1\dot{T}_{11}{}^{67} + \dot{Y}_1\dot{T}_{21}{}^{67} + \dot{Z}_1\dot{T}_{31}{}^{67})$$

$$+ C_{17}\ (\dot{X}_1\dot{T}_{11}{}^{8} + \dot{Y}_1\dot{T}_{21}{}^{8} + \dot{Z}_1\dot{T}_{31}{}^{8}$$

$$+ \dot{X}_1\dot{T}_{11}{}^{10} + \dot{Y}_1\dot{T}_{21}{}^{10} + \dot{Z}_1\dot{T}_{31}{}^{10})$$

$$- C_{18}\ (\dot{X}_1\dot{T}_{13}{}^{89} + \dot{Y}_1\dot{T}_{23}{}^{89} + \dot{Z}_1\dot{T}_{33}{}^{89} + \dot{X}_1\dot{T}_{13}{}^{1011}$$

$$+ \dot{Y}_1\dot{T}_{23}{}^{1011} + \dot{Z}_1\dot{T}_{33}{}^{1011})$$

$$+ C_{19}\ (\dot{T}_{13}{}^{1}\dot{T}_{13}{}^{2} + \dot{T}_{23}{}^{1}\dot{T}_{23}{}^{2} + \dot{T}_{33}{}^{1}\dot{T}_{33}{}^{2})$$

$$+ C_{20}\ (\dot{T}_{13}{}^{1}\dot{T}_{13}{}^{3} + \dot{T}_{23}{}^{1}\dot{T}_{23}{}^{3} + \dot{T}_{33}{}^{1}\dot{T}_{33}{}^{3})$$

$$+ C_{21}\ (\dot{T}_{13}{}^{1}\dot{T}_{11}{}^{3} + \dot{T}_{23}{}^{1}\dot{T}_{21}{}^{3} + \dot{T}_{33}{}^{1}\dot{T}_{31}{}^{3})$$

$$+ C_{22}\ (\dot{T}_{12}{}^{2}\dot{T}_{13}{}^{3} + \dot{T}_{23}{}^{2}\dot{T}_{23}{}^{3} + \dot{T}_{33}{}^{2}\dot{T}_{33}{}^{3})$$

$$+ C_{23}\ (\dot{T}_{13}{}^{2}\dot{T}_{11}{}^{3} + \dot{T}_{23}{}^{2}\dot{T}_{21}{}^{3} + \dot{T}_{33}{}^{2}\dot{T}_{31}{}^{3})$$

$$+ C_{24}\ (\dot{T}_{13}{}^{3}\dot{T}_{11}{}^{3} + \dot{T}_{23}{}^{3}\dot{T}_{21}{}^{3} + \dot{T}_{33}{}^{3}\dot{T}_{31}{}^{3})$$

$$- C_{25}\ (\dot{T}_{13}{}^{1}\dot{T}_{13}{}^{4} + \dot{T}_{23}{}^{1}\dot{T}_{23}{}^{4} + \dot{T}_{33}{}^{1}\dot{T}_{33}{}^{4}$$

$$+ \dot{T}_{13}{}^{1}\dot{T}_{13}{}^{6} + \dot{T}_{23}{}^{1}\dot{T}_{23}{}^{6} + \dot{T}_{33}{}^{1}\dot{T}_{33}{}^{6})$$

$$- C_{26}\ (\dot{T}_{13}{}^{2}\dot{T}_{13}{}^{4} + \dot{T}_{23}{}^{2}\dot{T}_{23}{}^{4} + \dot{T}_{33}{}^{2}\dot{T}_{33}{}^{4}$$

$$+ \dot{T}_{13}{}^{2}\dot{T}_{13}{}^{6} + \dot{T}_{23}{}^{2}\dot{T}_{23}{}^{6} + \dot{T}_{33}{}^{2}\dot{T}_{33}{}^{6})$$

$$+ C_{27}\ (\dot{T}_{12}{}^{2}\dot{T}_{13}{}^{4} + \dot{T}_{22}{}^{2}\dot{T}_{23}{}^{4} + \dot{T}_{32}{}^{2}\dot{T}_{33}{}^{4}$$

$$- \dot{T}_{12}{}^{2}\dot{T}_{13}{}^{6} - \dot{T}_{22}{}^{2}\dot{T}_{23}{}^{6} - \dot{T}_{32}{}^{2}\dot{T}_{33}{}^{6})$$

$$+ C_{28}\ (\dot{T}_{13}{}^{1}\dot{T}_{11}{}^{45} + \dot{T}_{23}{}^{1}\dot{T}_{21}{}^{45} + \dot{T}_{33}{}^{1}\dot{T}_{31}{}^{45}$$

$$+ \dot{T}_{13}{}^{1}\dot{T}_{11}{}^{67} + \dot{T}_{23}{}^{1}\dot{T}_{21}{}^{67} + \dot{T}_{33}{}^{1}\dot{T}_{31}{}^{67})$$

$$+ C_{29}\ (\dot{T}_{13}{}^{2}\dot{T}_{11}{}^{45} + \dot{T}_{23}{}^{2}\dot{T}_{21}{}^{45} + \dot{T}_{33}{}^{2}\dot{T}_{31}{}^{45}$$

$$+ \dot{T}_{13}{}^{2}\dot{T}_{11}{}^{67} + \dot{T}_{23}{}^{2}\dot{T}_{21}{}^{67} + \dot{T}_{33}{}^{2}\dot{T}_{31}{}^{67})$$

$$+ C_{30}\ (\dot{T}_{12}{}^{2}\dot{T}_{11}{}^{67} + \dot{T}_{22}{}^{2}\dot{T}_{21}{}^{67} + \dot{T}_{32}{}^{3}\dot{T}_{31}{}^{67}$$

$$- \dot{T}_{12}{}^{2}\dot{T}_{11}{}^{45} - \dot{T}_{22}{}^{2}\dot{T}_{21}{}^{45} - \dot{T}_{32}{}^{2}\dot{T}_{31}{}^{45})$$

$$
\begin{aligned}
& - C_{31}\ (\dot{T}_{13}{}^{4}\dot{T}_{11}{}^{45} + \dot{T}_{23}{}^{4}\dot{T}_{21}{}^{45} + \dot{T}_{33}{}^{4}\dot{T}_{31}{}^{45} \\
& \qquad + \dot{T}_{13}{}^{6}\dot{T}_{11}{}^{67} + \dot{T}_{23}{}^{6}\dot{T}_{21}{}^{67} + \dot{T}_{33}{}^{6}\dot{T}_{31}{}^{67}) \\
& - C_{32}\ (\dot{T}_{13}{}^{1}\dot{T}_{11}{}^{8} + \dot{T}_{23}{}^{1}\dot{T}_{21}{}^{8} + \dot{T}_{33}{}^{1}\dot{T}_{31}{}^{8} \\
& \qquad + \dot{T}_{13}{}^{1}\dot{T}_{11}{}^{10} + \dot{T}_{23}{}^{1}\dot{T}_{21}{}^{10} + \dot{T}_{33}{}^{1}\dot{T}_{31}{}^{10}) \\
& - C_{33}\ (\dot{T}_{12}{}^{1}\dot{T}_{11}{}^{8} + \dot{T}_{22}{}^{1}\dot{T}_{21}{}^{8} + \dot{T}_{32}{}^{1}\dot{T}_{31}{}^{8} \\
& \qquad - \dot{T}_{12}{}^{1}\dot{T}_{11}{}^{10} - \dot{T}_{22}{}^{1}\dot{T}_{21}{}^{10} - \dot{T}_{32}{}^{1}\dot{T}_{31}{}^{10}) \\
& + C_{34}\ (\dot{T}_{13}{}^{1}\dot{T}_{13}{}^{89} + \dot{T}_{23}{}^{1}\dot{T}_{23}{}^{89} + \dot{T}_{33}{}^{1}\dot{T}_{33}{}^{89} + \dot{T}_{13}{}^{1}\dot{T}_{13}{}^{1011} \\
& \qquad + \dot{T}_{23}{}^{1}\dot{T}_{23}{}^{1011} + \dot{T}_{33}{}^{1}\dot{T}_{33}{}^{1011}) \\
& + C_{35}\ (\dot{T}_{12}{}^{1}\dot{T}_{13}{}^{89} + \dot{T}_{22}{}^{1}\dot{T}_{23}{}^{89} + \dot{T}_{32}{}^{1}\dot{T}_{33}{}^{89} - \dot{T}_{12}{}^{1}\dot{T}_{13}{}^{1011} \\
& \qquad - \dot{T}_{22}{}^{1}\dot{T}_{23}{}^{1011} - \dot{T}_{32}{}^{1}\dot{T}_{33}{}^{1011}) \\
& - C_{36}\ (\dot{T}_{11}{}^{8}\dot{T}_{13}{}^{89} + \dot{T}_{21}{}^{8}\dot{T}_{23}{}^{89} + \dot{T}_{31}{}^{8}\dot{T}_{33}{}^{89} + \dot{T}_{11}{}^{10}\dot{T}_{13}{}^{1011} \\
& \qquad + \dot{T}_{21}{}^{10}\dot{T}_{23}{}^{1011} + \dot{T}_{31}{}^{10}\dot{T}_{33}{}^{1011}) \\
& + 1/2\ C_{37} \Big[\dot{\psi}_1{}^2\ (\sin^2\theta_1 + \cos^2\theta_1 \cos^2\phi_1) \\
& \qquad + \dot{\phi}_1{}^2 + \dot{\theta}_1{}^2 \sin^2\phi_1 \\
& \qquad - 2\dot{\psi}_1\dot{\theta}_1 \cos\theta_1 \sin\phi_1 \cos\phi_1
\end{aligned}
$$

$$- 2\dot{\psi}_1\dot{\phi}_1 \sin \theta_1 \Big]$$

$$+ 1/2\ C_{38} \Big[\dot{\psi}_1{}^2 \cos^2 \theta_1 \sin^2 \phi_1 + \dot{\theta}_1{}^2 \cos^2 \phi_1$$

$$+ 2\dot{\psi}_1\dot{\theta}_1 \cos \theta_1 \sin \phi_1 \cos \phi_1\Big]$$

$$+ 1/2\ C_{39}\ (\dot{\psi}_2{}^2 \sin^2 \theta_2 + \dot{\phi}_2{}^2 - 2\dot{\psi}_2\dot{\phi}_2 \sin \theta_2)$$

$$+ 1/2\ C_{40}\ (\dot{\psi}_2{}^2 \cos^2 \theta_2 + \dot{\theta}_2{}^2)$$

$$+ 1/2\ C_{41} \Big[\dot{\psi}_3{}^2\ (\sin^2 \theta_3 + \cos^2 \theta_3 \sin^2 \phi_3)$$

$$+ \dot{\phi}_3{}^2 + \dot{\theta}_3{}^2 \cos^2 \phi_3$$

$$+ 2\dot{\psi}_3\dot{\theta}_3 T_{32}{}^3 \cos \phi_3$$

$$+ 2\dot{\psi}_3\dot{\phi}_3 T_{31}{}^3 \Big]$$

$$+ 1/2\ C_{42} \Big[\dot{\psi}_3{}^2 \cos^2 \theta_3 \cos^2 \phi_3 + \dot{\theta}_3{}^2 \sin^2 \phi_3$$

$$- 2\dot{\psi}_3\theta_3 T_{33}{}^3 \sin \phi_3\Big]$$

$$+ 1/2\ C_{43} \Big[\dot{\psi}_4{}^2\ (\sin^2 \theta_4 + \cos^2 \theta_4 \sin^2 \phi_4)$$

$$+ \dot{\phi}_4{}^2 + \dot{\theta}_4{}^2 \cos^2 \phi_4$$

$$+ 2\dot{\psi}_4\dot{\theta}_4 T_{32}{}^4 \cos \phi_4$$

$$- 2\dot{\psi}_4\dot{\phi}_4 \sin \theta_4\Big]$$

$$+ 1/2\ C_{44}\ (\dot{\psi}_4{}^2 \cos^2 \theta_4 \cos^2 \phi_4 + \dot{\theta}_4{}^2 \sin^2 \phi_4$$

$$- 2\dot{\psi}_4\dot{\theta}_4 \cos \theta_4 \sin \phi_4 \cos \phi_4)$$

$$+ 1/2\ C_{45} \Big[\dot{\psi}_4^{\,2} (\sin^2 \theta_4 \sin^2 \alpha_5 + \cos^2 \theta_4 \cos^2 \phi_4 \cos^2 \alpha_5$$

$$+ 2 \sin \theta_4 \cos \theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5)$$

$$+ \dot{\theta}_4^{\,2} \sin^2 \phi_4 \cos^2 \alpha_5$$

$$+ \dot{\phi}_4^{\,2} \sin^2 \alpha_5$$

$$- 2\dot{\psi}_4\dot{\theta}_4 \sin \theta_4 \sin \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$- 2\dot{\psi}_4\dot{\theta}_4 \cos \theta_4 \cos \phi_4 \sin \phi_4 \cos^2 \alpha_5$$

$$- 2\dot{\psi}_4\dot{\phi}_4 \sin \theta_4 \sin^2 \alpha_5$$

$$- 2\dot{\psi}_4\dot{\phi}_4 \cos \theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ 2\dot{\theta}_4\dot{\phi}_4 \sin \phi_4 \sin \alpha_5 \cos \alpha_5\Big]$$

$$+ 1/2\ C_{46} \Big[\dot{\psi}_4^{\,2} (\cos^2 \theta_4 \sin^2 \phi_4$$

$$+ \sin^2 \theta_4 \cos^2 \alpha_5 + \cos^2 \theta_4 \cos^2 \phi_4 \sin^2 \alpha_5$$

$$- 2 \sin \theta_4 \cos \theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5)$$

$$+ \dot{\theta}_4^{\,2} (\cos^2 \phi_4 + \sin^2 \phi_4 \sin^2 \alpha_5) + \dot{\phi}_4^{\,2} \cos^2 \alpha_5 + \dot{\alpha}_5^{\,2}$$

$$+ 2\dot{\psi}_4\dot{\theta}_4 \cos \theta_4 \sin \phi_4 \cos \phi_4 \cos^2 \alpha_5$$

$$+ 2\dot{\psi}_4\dot{\theta}_4 \sin \theta_4 \sin \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$- 2\dot{\psi}_4\dot{\phi}_4 \sin \theta_4 \cos^2 \alpha_5$$

$$+ 2\dot{\psi}_4\dot{\phi}_4 \cos \theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ 2\dot{\psi}_4\dot{\alpha}_5 \cos \theta_4 \sin \phi_4$$

$$- 2\dot{\theta}_4\dot{\phi}_4 \sin\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ 2\dot{\theta}_4\dot{\alpha}_5 \cos\phi_4\Big]$$

$$+ 1/2\ C_{43}\Big[\dot{\psi}_6^{\ 2}\ (\sin^2\theta_6 + \cos^2\theta_6 \sin^2\phi_6) + \dot{\phi}_6^{\ 2} + \dot{\theta}_6^{\ 2} \cos^2\phi_6$$

$$+ 2\ \dot{\psi}_6\dot{\theta}_6 \cos\theta_6 \sin\phi_6 \cos\phi_6 - 2\dot{\psi}_5\dot{\phi}_6 \sin\theta_6\Big]$$

$$+ 1/2\ C_{44}\Big[\dot{\psi}_6^{\ 2} \cos^2\theta_6 \cos^2\phi_6 + \dot{\theta}_6^{\ 2} \sin^2\phi_6$$

$$- 2\dot{\psi}_6\dot{\theta}_6 \cos\theta_6 \sin\phi_6 \cos\phi_6\Big]$$

$$+ 1/2\ C_{45}\Big[\dot{\psi}_6^{\ 2}\ (\sin^2\theta_6 \sin^2\alpha_7 + \cos^2\theta_6 \cos^2\phi_6 \cos^2\alpha_7$$

$$+ 2\sin\theta_6 \cos\theta_6 \cos\phi_6 \sin\alpha_7 \cos\alpha_7)$$

$$+ \dot{\theta}_6^{\ 2} \sin^2\phi_6 \cos^2\alpha_7 + \dot{\phi}_6^{\ 2} \sin^2\alpha_7$$

$$- 2\dot{\psi}_6\dot{\theta}_6 \sin\theta_6 \sin\phi_6 \sin\alpha_7 \cos\alpha_7$$

$$- 2\dot{\psi}_6\dot{\theta}_6 \cos\theta_6 \sin\phi_6 \cos\phi_6 \cos^2\alpha_7$$

$$- 2\dot{\psi}_6\dot{\phi}_6 \sin\theta_6 \sin^2\alpha_7$$

$$- 2\dot{\psi}_6\dot{\phi}_6 \cos\theta_6 \cos\phi_6 \sin\alpha_7 \cos\alpha_7$$

$$+ 2\dot{\theta}_6\dot{\phi}_6 \sin\phi_6 \sin\alpha_7 \cos\alpha_7\Big]$$

$$+ 1/2\ C_{46}\Big[\dot{\psi}_6^{\ 2}\ (\cos^2\theta_6 \sin^2\phi_6 + \sin^2\theta_6 \cos^2\alpha_7$$

$$+ \cos^2\theta_6 \cos^2\phi_6 \sin^2\alpha_7$$

$$- 2\sin\theta_6 \cos\theta_6 \cos\phi_6 \sin\alpha_7 \cos\alpha_7)$$

$$+ \dot{\theta}_6^{\ 2}\ (\cos^2\phi_6 + \sin^2\phi_6 \sin^2\alpha_7) + \dot{\phi}_6^{\ 2} \cos^2\alpha_7 + \dot{\alpha}_7^{\ 2}$$

$$+ 2\dot{\psi}_6\dot{\theta}_6 \cos\theta_6 \sin\phi_5 \cos\phi_6 \cos^2\alpha_7$$

$$+ 2\dot{\psi}_6\dot{\theta}_6 \sin\theta_6 \sin\phi_6 \sin\alpha_7 \cos\alpha_7$$

$$- 2\dot{\psi}_6\dot{\phi}_6 \sin\theta_6 \cos^2\alpha_7$$

$$+ 2\dot{\psi}_6\dot{\phi}_6 \cos\theta_6 \cos\phi_6 \sin\alpha_7 \cos\alpha_7$$

$$+ 2\dot{\psi}_6\dot{\alpha}_7 \cos\theta_6 \sin\phi_6$$

$$- 2\dot{\theta}_6\dot{\phi}_6 \sin\phi_6 \sin\alpha_7 \cos\alpha_7$$

$$+ 2\dot{\theta}_6\dot{\alpha}_7 \cos\phi_6\Big]$$

$$+ 1/2\ C_{47}\ (\dot{\psi}_8^{\,2} \sin^2\theta_8 + \dot{\phi}_8^{\,2} - 2\dot{\psi}_8\dot{\phi}_8 \sin\theta_8)$$

$$+ 1/2\ C_{48}\ (\dot{\psi}_8^{\,2} \cos^2\theta_8 + \dot{\theta}_8^{\,2})$$

$$+ 1/2\ C_{49}\ \Big[\dot{\psi}_8^{\,2}\ (\sin^2\theta_8 \sin^2\alpha_9 + \cos^2\theta_8 \cos^2\phi_8 \cos^2\alpha_9$$

$$+ \cos^2\theta_8 \sin^2\phi_8$$

$$- \quad \sin\theta_8 \cos\theta_8 \cos\phi_8 \sin\alpha_9 \cos\alpha_9)$$

$$+ \dot{\theta}_8^{\,2}\ (\sin^2\phi_8 \cos^2\alpha_9 + \cos^2\phi_8) + \dot{\phi}_8 2 \sin^2\alpha_9$$

$$+ \dot{\alpha}_9^{\,2} + 2\dot{\psi}_8\dot{\theta}_8 \sin\theta_8 \sin\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$+ 2\dot{\psi}_8\dot{\theta}_8 \cos\theta_8 \sin\phi_8 \cos\phi_8 \sin^2\alpha_9$$

$$- 2\dot{\psi}_8\dot{\phi}_8 \sin\theta_8 \sin^2\alpha_9$$

$$+ 2\dot{\psi}_8\dot{\phi}_8 \cos\theta_8 \cos\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$+ 2\dot{\psi}_8\dot{\alpha}_9 \cos\theta_8 \sin\phi_8 - 2\theta_8\phi_8 \sin\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$+ 2\dot{\theta}_8\dot{\alpha}_9 \cos\phi_8 \Big]$$

$$+ 1/2\ C_{50} \Big[\dot{\psi}_8^{\,2} (\cos^2\theta_8 \cos^2\phi_8 \sin^2\alpha_9 + \sin^2\theta_8 \cos^2\alpha_9$$

$$+ 2 \sin\theta_8 \cos\theta_8 \cos\phi_8 \sin\alpha_9 \cos\alpha_9)$$

$$+ \dot{\theta}_8^{\,2} \sin^2\phi_8 \sin^2\alpha_9 + \dot{\phi}_8^{\,2} \cos^2\alpha_9$$

$$- 2\dot{\psi}_8\dot{\theta}_8 \cos\theta_9 \sin\phi_8 \cos\phi_8 \sin^2\alpha_9$$

$$- 2\dot{\psi}_8\dot{\theta}_8 \sin\theta_8 \sin\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$- 2\dot{\psi}_8\dot{\phi}_8 \sin\phi_8 \cos^2\alpha_9 + 2\dot{\theta}_8\dot{\phi}_8 \sin\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$- 2\dot{\psi}_8\dot{\phi}_8 \cos\theta_8 \cos\phi_8 \sin\alpha_9 \cos\alpha_9 \Big]$$

$$+ \dot{\theta}_{10}^{\,2} \sin^2\phi_{10} \sin^2\alpha_{11} + \dot{\phi}_{10}^{\,2} \cos^2\alpha_{11}$$

$$- 2\dot{\psi}_{10}\dot{\theta}_{10} \cos\theta_{10} \sin\phi_{10} \cos\phi_{10} \sin^2\alpha_{11}$$

$$- 2\dot{\psi}_{10}\dot{\theta}_{10} \sin\theta_{10} \sin\phi_{10} \sin\alpha_{11} \cos\alpha_{11}$$

$$- 2\dot{\psi}_{10}\dot{\phi}_{10} \cos\theta_{10} \cos\phi_{10} \sin\alpha_{11} \cos\alpha_{11}$$

$$- 2\dot{\psi}_{10}\dot{\phi}_{10} \sin\theta_{10} \cos^2\alpha_{11}$$

$$+ 2\dot{\theta}_{10}\dot{\phi}_{10} \sin\phi_{10} \sin\alpha_{11} \cos\alpha_{11} \Big]$$

$$+ 1/2\ C_{47}\ (\dot{\psi}_{10}{}^{2} \sin^{2} \theta_{10} + \dot{\phi}_{10}{}^{2} - 2\dot{\psi}_{10}\dot{\phi}_{10} \sin \theta_{10})$$

$$+ 1/2\ C_{48}\ (\dot{\psi}_{10}{}^{2} \cos^{2} \theta_{10} + \dot{\theta}_{10}{}^{2})$$

$$+ 1/2\ C_{49} \Big[\dot{\psi}_{10}{}^{2}\ (\sin^{2} \theta_{10} \sin^{2} \alpha_{11} + \cos^{2} \theta_{10} \cos^{2} \phi_{10} \cos^{2} \alpha_{11}$$

$$+ \cos^{2} \theta_{10} \sin^{2} \phi_{10}$$

$$- 2 \sin \theta_{10} \cos \theta_{10} \cos \phi_{10} \sin \alpha_{11} \cos \alpha_{11})$$

$$+ \dot{\theta}_{10}{}^{2}\ (\sin^{2} \phi_{10} \cos^{2} \alpha_{11} + \cos^{2} \phi_{10}) + \dot{\phi}_{10}{}^{2} \sin^{2} \alpha_{11}$$

$$+ \dot{\alpha}_{11}{}^{2} + 2\dot{\psi}_{10}\dot{\theta}_{10} \sin \theta_{10} \sin \phi_{10} \sin \alpha_{11} \cos \alpha_{11}$$

$$+ 2\dot{\psi}_{10}\dot{\theta}_{10} \cos \theta_{10} \sin \phi_{10} \cos \phi_{10} \sin^{2} \alpha_{11}$$

$$- 2\dot{\psi}_{10}\dot{\phi}_{10} \sin \theta_{10} \sin^{2} \alpha_{11}$$

$$+ 2\dot{\psi}_{10}\dot{\phi}_{10} \cos \theta_{10} \cos \phi_{10} \sin \alpha_{11} \cos \alpha_{11}$$

$$+ 2\dot{\psi}_{10}\dot{\alpha}_{11} \cos \theta_{10} \sin \phi_{10}$$

$$- 2\dot{\theta}_{10}\dot{\alpha}_{10} \sin \phi_{10} \sin \alpha_{11} \cos \alpha_{11}$$

$$+ 2\dot{\theta}_{10}\dot{\alpha}_{11} \cos \phi_{10}\Big]$$

$$+ 1/2\ C_{50} \Big[\dot{\psi}_{10}{}^{2}\ (\cos^{2} \theta_{10} \cos^{2} \phi_{10} \sin^{2} \alpha_{11}$$

$$+ \sin^{2} \theta_{10} \cos^{2} \alpha_{11}$$

$$+ 2 \sin \theta_{10} \cos \theta_{10} \cos \phi_{10} \sin \alpha_{11} \cos \alpha_{11})$$

APPENDIX C

OCCUPANT POTENTIAL ENERGY

Potential Energy:

$$V_\rho = M_1Z_1 + M_2Z_2 + M_3Z_3 + \ldots + M_{11}Z_{11}$$

where

$$Z_1 = Z_1$$

$$Z_2 = Z_1 + \bar{\rho}_1 T_{33}{}^1 + \rho_2 T_{33}{}^2$$

$$Z_3 = Z_1 + \bar{\rho}_1 T_{33}{}^1 + L_2 T_{33}{}^2 + \rho_3 T_{33}{}^3 + L_c T_{31}{}^3$$

$$Z_4 = Z_1 + \bar{\rho}_1 T_{33}{}^1 + L_2 T_{33}{}^2 - L_s T_{32}{}^2 - \rho_4 T_{33}{}^4$$

$$Z_5 = Z_1 + \bar{\rho}_1 T_{33}{}^1 + L_2 T_{33}{}^2 - L_s T_{32}{}^2 - L_4 T_{33}{}^4 + \rho_5 T_{31}{}^5$$

$$Z_6 = Z_1 + \bar{\rho}_1 T_{33}{}^1 + L_2 T_{33}{}^2 + L_s T_{32}{}^2 - \rho_4 T_{33}{}^6$$

$$Z_7 = Z_1 + \bar{\rho}_1 T_{33}{}^1 + L_2 T_{33}{}^2 + L_s T_{32}{}^2 - L_4 T_{33}{}^6 + \rho_5 T_{31}{}^7$$

$$Z_8 = Z_1 - \rho_1 T_{33}{}^1 - L_H T_{32}{}^1 + \rho_8 T_{31}{}^8$$

$$Z_9 = Z_1 - \rho_1 T_{33}{}^1 - L_H T_{32}{}^1 + L_8 T_{31}{}^8 - \rho_9 T_{33}{}^9$$

$$Z_{10} = Z_1 - \rho_1 T_{33}{}^1 + L_H T_{32}{}^1 + \rho_8 T_{31}{}^{10}$$

$$Z_{11} = Z_1 - \rho_1 T_{33}{}^1 + L_H T_{32}{}^1 + L_8 T_{31}{}^{10} - \rho_9 T_{33}{}^{11}$$

APPENDIX D

[A], {B}

Below are listed the non-zero elements of the matrix [A] and the vector {B}, derived from the kinetic energy derivatives. Because [A] is symmetric, only the upper triangular part is given to save space, i.e.

$$A_{ij} \text{ where } j \leq i$$

$A\ (1,1) = C_1$

$A\ (1,4) = C_{11}T_{23}{}^1$

$A\ (1,5) = C_{11}T_{11}{}^1 \cos \phi_1$

$A\ (1,6) = - C_{11}T_{12}{}^1$

$A\ (1,7) = - C_{12}T_{23}{}^2$

$A\ (1,8) = C_{12}T_{11}{}^2 \cos \phi_2$

$A\ (1,9) = - C_{12}T_{12}{}^2$

$A\ (1,10) = - C_{13}T_{23}{}^3 - C_{14}T_{21}{}^3$

$A\ (1,11) = (C_{13}T_{33}{}^3 + C_{14}T_{13}{}^3) \cos \psi_3$

$A\ (1,12) = - C_{13}T_{12}{}^3$

$A\ (1,13) = C_{15}T_{23}{}^4 - C_{16}T_{21}{}^5$

$A\ (1,14) = (C_{16}T_{31}{}^5 - C_{15}T_{33}{}^4) \cos \psi_4$

$A\ (1,15) = (C_{15} + C_{16} \cos \alpha_5)\ T_{12}{}^4$

$A\ (1,16) = C_{16}T_{13}{}^5$

$$A\ (1,17) = C_{15}T_{23}{}^{6} - C_{16}T_{21}{}^{7}$$

$$A\ (1,18) = (C_{16}T_{31}{}^{7} - C_{15}T_{33}{}^{6}) \quad \cos \psi_6$$

$$A\ (1,19) = (C_{15} + C_{16} \cos \alpha_7)\ T_{12}{}^{6}$$

$$A\ (1,20) = C_{16}T_{13}{}^{7}$$

$$A\ (1,21) = -\ C_{17}T_{21}{}^{8} + C_{18}T_{23}{}^{9}$$

$$A\ (1,22) = -\ (C_{17} \sin \theta_8 + C_{18}T_{33}{}^{9}) \cos \psi_8$$

$$A\ (1,23) = C_{18}T_{12}{}^{8} \sin \alpha_9$$

$$A\ (1,24) = -\ C_{18}T_{11}{}^{9}$$

$$A\ (1,25) = -\ C_{17}T_{21}{}^{10} + C_{18}T_{23}{}^{11}$$

$$A\ (1,26) = -\ (C_{17} \sin \theta_{10} + C_{18}T_{33}{}^{11}) \cos \psi_{10}$$

$$A\ (1,27) = C_{18}T_{12}{}^{10} \sin \alpha_{11}$$

$$A\ (1,28) = -\ C_{18}T_{11}{}^{11}$$

$$A\ (2,2) = C_1$$

$$A\ (2,4) = C_{11}T_{13}{}^{1}$$

$$A\ (2,5) = C_{11}T_{21}{}^{1} \cos \phi_1$$

$$A\ (2,6) = -\ C_{11}T_{22}{}^{1}$$

$$A\ (2,7) = C_{12}T_{13}{}^{2}$$

$$A\ (2,8) = C_{12}T_{21}^{\ 2}\cos\phi_2$$

$$A\ (2,9) = -\ C_{12}T_{22}^{\ 2}$$

$$A\ (2,10) = C_{13}T_{13}^{\ 3} + C_{14}T_{11}^{\ 3}$$

$$A\ (2,11) = (C_{13}T_{33}^{\ 3} + C_{14}T_{31}^{\ 3})\sin\psi_3$$

$$A\ (2,12) = -\ C_{13}T_{22}^{\ 3}$$

$$A\ (2,13) = -\ C_{15}T_{13}^{\ 4} + C_{16}T_{11}^{\ 5}$$

$$A\ (2,14) = (C_{16}T_{31}^{\ 5} - C_{15}T_{33}^{\ 4})\sin\psi_4$$

$$A\ (2,15) = (C_{15} + C_{16}\cos\alpha_5)\ T_{22}^{\ 4}$$

$$A\ (2,16) = C_{16}T_{23}^{\ 5}$$

$$A\ (2,17) = -\ C_{15}T_{13}^{\ 6} + C_{16}T_{11}^{\ 7}$$

$$A\ (2,18) = (C_{16}T_{31}^{\ 7} - C_{15}T_{33}^{\ 6})\sin\psi_6$$

$$A\ (2,19) = (C_{15} + C_{16}\cos\alpha_7)\ T_{22}^{\ 6}$$

$$A\ (2,20) = C_{16}T_{23}^{\ 7}$$

$$A\ (2,21) = C_{17}T_{11}^{\ 8} - C_{18}T_{13}^{\ 9}$$

$$A\ (2,22) = -\ (C_{17}\sin\theta_8 + C_{18}T_{33}^{\ 9})\sin\psi_8$$

$$A\ (2,23) = C_{18}T_{22}^{\ 8}\sin\alpha_9$$

$$A\ (2,24) = -\ C_{18}T_{21}^{\ 9}$$

$$A\ (2,25) = C_{17}T_{11}^{10} - C_{18}T_{13}^{11}$$

$$A\ (2,26) = -\ (C_{17} \sin \theta_{10} + C_{18}T_{33}^{11}) \sin \psi_{10}$$

$$A\ (2,27) = C_{18}T_{22}^{10} \sin \alpha_{11}$$

$$A\ (2,28) = -\ C_{18}T_{21}^{11}$$

$$A\ (3,3) = C_1$$

$$A\ (3,5) = C_{11}T_{31}^{1} \cos \phi_1$$

$$A\ (3,6) = -\ C_{11}T_{32}^{1}$$

$$A\ (3,8) = C_{12}T_{31}^{2} \cos \phi_2$$

$$A\ (3,9) = -\ C_{12}T_{32}^{2}$$

$$A\ (3,11) = C_{13}T_{31}^{3} \cos \phi_3 - C_{14} \cos \theta_3$$

$$A\ (3,12) = -C_{13}T_{32}^{3}$$

$$A\ (3,14) = C_{16}(\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5) - C_{15}T_{31}^{4} \cos \phi_4$$

$$A\ (3,15) = (C_{15} + C_{16} \cos \alpha_5)\ T_{32}^{4}$$

$$A\ (3,16) = C_{16}T_{33}^{5}$$

$$A\ (3,18) = C_{16}\ (\sin \theta_6 \cos \phi_6 \cos \alpha_7 - \cos \theta_6 \sin \alpha_7) - C_{15}T_{31}^{6} \cos \phi_6$$

$$A\ (3,19) = (C_{15} + C_{16} \cos \alpha_7)\ T_{32}^{\ 6}$$

$$A\ (3,20) = C_{16} T_{33}^{\ 7}$$

$$A\ (3,22) = - C_{17} \cos \theta_8$$

$$- C_{18} (\cos \theta_8 \cos \alpha_9 - \sin \theta_8 \cos \phi_8 \sin \alpha_9)$$

$$A\ (3,23) = C_{18} T_{32}^{\ 8} \sin \alpha_9$$

$$A\ (3,24) = - C_{18} T_{31}^{\ 9}$$

$$A\ (3,26) = - C_{17} \cos \theta_{10}$$

$$- C_{18} (\cos \theta_{10} \cos \alpha_{11} - \sin \theta_{10} \cos \phi_{10} \sin \alpha_{11})$$

$$A\ (3,27) = C_{18} T_{32}^{\ 10} \sin \alpha_{11}$$

$$A\ (3,28) = - C_{18} T_{31}^{\ 11}$$

$$A\ (4,4) = (C_2 + C_6) \sin^2 \phi_1 + C_{37} \sin^2 \theta_1$$

$$+ (C_2 \sin^2 \theta_1 + C_6 + C_{37} \cos^2 \theta_1) \cos^2 \phi_1$$

$$A\ (4,5) = (C_2 - C_6 - C_{37} + C_{38})\ T_{32}^{\ 1} \cos \phi_1$$

$$A\ (4,6) = (C_2 + C_6 + C_{37})\ T_{31}^{\ 1}$$

$$A\ (4,7) = C_{19}\ (T_{13}^{\ 1} T_{13}^{\ 2} + T_{23}^{\ 1} T_{23}^{\ 2})$$

$$A\ (4,8) = C_{19} \cos \phi_2\ (T_{13}^{\ 1} T_{21}^{\ 2} - T_{23}^{\ 1} T_{11}^{\ 2})$$

$$A\ (4,9) = C_{19} (T_{23}^{\ 1} T_{12}^{\ 2} - T_{13}^{\ 1} T_{22}^{\ 2})$$

$$A\ (4,10) = C_{20}\ (T_{13}^{1}T_{13}^{3} + T_{23}^{1}T_{23}^{3})$$

$$+ C_{21}\ (T_{13}^{1}T_{11}^{3} + T_{23}^{1}T_{21}^{3})$$

$$A\ (4,11) = (C_{20}T_{33}^{3} + C_{21}T_{31}^{3})\ T_{13}^{1} \sin \psi_3$$

$$- (C_{20}T_{33}^{3} + C_{21}T_{31}^{3})\ T_{23}^{1} \cos \psi_3$$

$$A\ (4,12) = C_{20}\ (T_{23}^{1}T_{12}^{3} - T_{13}^{1}T_{22}^{3})$$

$$A\ (4,13) = -\ C_{25}\ (T_{23}^{1}T_{23}^{4} + T_{13}^{1}T_{13}^{4})$$

$$+ C_{28}\ (T_{23}^{1}T_{21}^{5} + T_{13}^{1}T_{11}^{5})$$

$$A\ (4,14) = (C_{25}T_{33}^{4} - C_{28}T_{31}^{5})\ (T_{23}^{1} \cos \psi_4 - T_{13}^{1} \sin \psi_4)$$

$$A\ (4,15) = (C_{25} + C_{28} \cos \alpha_5)\ (T_{13}^{1}T_{22}^{4} - T_{23}^{1}T_{12}^{4})$$

$$A\ (4,16) = C_{28}\ (T_{13}^{1}T_{23}^{5} - T_{23}^{1}T_{13}^{5})$$

$$A\ (4,17) = -\ C_{25}\ (T_{13}^{1}T_{13}^{6} + T_{23}^{1}T_{23}^{6})$$

$$+ C_{28}\ (T_{23}^{1}T_{21}^{7} + T_{13}^{1}T_{11}^{7})$$

$$A\ (4,18) = (C_{25}T_{33}^{6} - C_{28}T_{31}^{7})\ (T_{23}^{1} \cos \psi_6 - T_{13}^{1} \sin \psi_6)$$

$$A\ (4,19) = (C_{25} + C_{28} \cos \alpha_7)\ (T_{13}^{1}T_{22}^{6} - T_{23}^{1}T_{12}^{6})$$

$$A\ (4,20) = C_{28}\ (T_{13}^{1}T_{23}^{7} - T_{23}^{1}T_{13}^{7})$$

$$A\ (4,21) = C_{34}\ (T_{13}^{1}T_{13}^{9} + T_{23}^{1}T_{23}^{9})$$

$$+ C_{35}\ (T_{12}^{1}T_{13}^{9} + T_{22}^{1}T_{23}^{9})$$

$$- C_{32}\ (T_{13}^{1}T_{11}^{8} + T_{23}^{1}T_{21}^{8})$$

$$- C_{33}\ (T_{12}^{1}T_{11}^{8} + T_{22}^{1}T_{21}^{8})$$

$$A\ (4,22) = (C_{34}T_{33}^{9} - C_{32}T_{31}^{8})\ (T_{13}^{1} \sin \psi_8 - T_{23}^{1} \cos \psi_8)$$

$$+ (C_{35}T_{33}^{9} - C_{33}T_{31}^{8})\ (T_{12}^{1} \sin \psi_8 - T_{22}^{1} \cos \psi_8)$$

$$A\ (4,23) = \Big[C_{34}\ (T_{23}^{1}T_{12}^{8} - T_{13}^{1}T_{22}^{8})$$

$$+ C_{35}\ (T_{22}^{1}T_{12}^{8} - T_{12}^{1}T_{22}^{8})\Big] \sin \alpha_9$$

$$A\ (4,24) = C_{34}\ (T_{13}^{1}T_{21}^{9} - T_{23}^{1}T_{11}^{9})$$

$$+ C_{35}\ (T_{12}^{1}T_{21}^{9} - T_{22}^{1}T_{11}^{9})$$

$$A\ (4,25) = C_{34}\ (T_{13}^{1}T_{13}^{11} + T_{23}^{1}T_{23}^{11})$$

$$- C_{35}\ (T_{12}^{1}T_{13}^{11} + T_{22}^{1}T_{23}^{11})$$

$$- C_{32}\ (T_{13}^{1}T_{11}^{10} + T_{23}^{1}T_{21}^{10})$$

$$+ C_{33}\ (T_{12}^{1}T_{11}^{10} + T_{22}^{1}T_{21}^{10})$$

$$A\ (4,26) = (C_{34}T_{33}^{11} - C_{32}T_{31}^{10})\ (T_{13}^{1} \sin \psi_{10} - T_{23}^{1} \cos \psi_{10})$$

$$- (C_{35}T_{33}^{11} - C_{33}T_{31}^{10})\ (T_{12}^{1} \sin \psi_{10} - T_{22}^{1} \cos \psi_{10})$$

$$A\ (4,27) = \Big[C_{35}\ (T_{12}^{1}T_{22}^{10} - T_{22}^{1}T_{12}^{10})$$

$$- C_{34}\ (T_{13}^{1}T_{22}^{10} - T_{23}^{1}T_{12}^{10})\Big] \sin \alpha_{11}$$

$$A\ (4,28) = C_{34}\ (T_{13}^{1}T_{21}^{11} - T_{23}^{1}T_{11}^{11})$$

$$- C_{35} \, (T_{12}{}^{1} T_{21}{}^{11} - T_{22}{}^{1} T_{11}{}^{1})$$

$$A\,(5,5) = (C_2 + C_{38}) \cos^2 \phi_1 + (C_6 + C_{37}) \sin^2 \phi_1$$

$$A\,(5,7) = C_{19} \, (T_{21}{}^{1} T_{13}{}^{2} - T_{11}{}^{1} T_{23}{}^{2}) \cos \phi_1$$

$$A\,(5,8) = C_{19} \cos \phi_1 \cos \phi_2 \, (T_{11}{}^{1} T_{11}{}^{2} + T_{21}{}^{1} T_{21}{}^{2}$$

$$+ T_{31}{}^{1} T_{31}{}^{2})$$

$$A\,(5,9) = - C_{19} \cos \phi_1 \, (T_{11}{}^{1} T_{12}{}^{2} + T_{21}{}^{1} T_{22}{}^{2} + T_{31}{}^{1} T_{32}{}^{2})$$

$$A\,(5,10) = \Big[(C_{20} T_{13}{}^{3} + C_{21} T_{11}{}^{3}) \, T_{21}{}^{1}$$

$$- (C_{20} T_{23}{}^{3} + C_{21} T_{21}{}^{3}) \, T_{11}{}^{1} \Big] \cos \phi_1$$

$$A\,(5,11) = \Big[C_{20} \cos \phi_3 \, (T_{11}{}^{1} T_{11}{}^{3} + T_{21}{}^{1} T_{21}{}^{3} + T_{31}{}^{1} T_{31}{}^{3})$$

$$- C_{21} \, \{ (T_{11}{}^{1} \cos \psi_3 + T_{21}{}^{1} \sin \psi_3) \sin \theta_3$$

$$+ T_{31}{}^{1} \cos \theta_3 \} \Big] \cos \phi_1$$

$$A\,(5,12) = - C_{20} \cos \phi_1 \, (T_{11}{}^{1} T_{12}{}^{3} + T_{21}{}^{1} T_{22}{}^{3} + T_{31}{}^{1} T_{32}{}^{3})$$

$$A\,(5,13) = \Big[C_{25} \, (T_{11}{}^{1} T_{23}{}^{4} - T_{21}{}^{1} T_{13}{}^{4})$$

$$- C_{28} \, (T_{11}{}^{1} T_{21}{}^{5} - T_{21}{}^{1} T_{11}{}^{5}) \Big] \cos \phi_1$$

$$A\,(5,14) = - C_{25} \, (T_{11}{}^{1} T_{11}{}^{4} + T_{21}{}^{1} T_{21}{}^{4} + T_{31}{}^{1} T_{31}{}^{4}) \cos \phi_4$$

$$+ C_{28} \Big[(T_{11}{}^{1} \cos \psi_4 + T_{21}{}^{1} \sin \psi_4) \, T_{31}{}^{5}$$

$$+ T_{31}{}^{1} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5) \Big] \cos \phi_1$$

$$A\ (5,15) = (C_{25} + C_{28} \cos \alpha_5)\ ({T_{11}}^1 {T_{12}}^4 + {T_{21}}^1 {T_{22}}^4$$

$$+ {T_{31}}^1 {T_{32}}^4) \cos \phi_1$$

$$A\ (5,16) = C_{28} \cos \phi_1\ ({T_{11}}^1 {T_{13}}^5 + {T_{21}}^1 {T_{23}}^5 + {T_{31}}^1 {T_{33}}^5)$$

$$A\ (5,17) = \Big[C_{25}\ ({T_{11}}^1 {T_{23}}^3 - {T_{21}}^1 {T_{13}}^6)$$

$$- C_{28}\ ({T_{11}}^1 {T_{21}}^7 - {T_{21}}^1 {T_{11}}^7) \Big] \cos \phi_1$$

$$A\ (5,18) = + \Big\{ C_{28} \Big[({T_{11}}^1 \cos \psi_6 + {T_{21}}^1 \sin \psi_6)\ {T_{31}}^7$$

$$+ {T_{31}}^1 (\sin \theta_6 \cos \phi_6 \cos \alpha_7 - \cos \theta_6 \sin \alpha_7) \Big]$$

$$- C_{25}\ ({T_{11}}^1 {T_{11}}^6 + {T_{21}}^1 {T_{21}}^6 + {T_{31}}^1 {T_{31}}^6) \cos \phi_6 \Big\} \cos \phi_1$$

$$A\ (5,19) = (C_{25} + C_{28} \cos \alpha_7)\ ({T_{11}}^1 {T_{12}}^6 + {T_{21}}^1 {T_{22}}^6$$

$$+ {T_{31}}^1 {T_{32}}^6) \cos \phi_1$$

$$A\ (5,20) = C_{28} \cos \phi_1\ ({T_{11}}^1 {T_{13}}^7 + {T_{21}}^1 {T_{23}}^7 + {T_{31}}^1 {T_{33}}^7)$$

$$A\ (5,21) = (C_{32} \cos \phi_1 + C_{33} \sin \phi_1)\ ({T_{11}}^1 {T_{21}}^8 - {T_{21}}^1 {T_{11}}^8)$$

$$- (C_{34} \cos \phi_1 + C_{35} \sin \phi_1)\ ({T_{11}}^1 {T_{23}}^9 - {T_{21}}^1 {T_{13}}^9)$$

$$A\ (5,22) = (C_{32} \cos \phi_1 + C_{33} \sin \phi_1)\ \Big[({T_{11}}^1 \cos \psi_8$$

$$+ {T_{21}}^1 \sin \psi_8) \sin \theta_8 + {T_{31}}^1 \cos \theta_8 \Big]$$

$$+ (C_{34} \cos \phi_1 + C_{35} \sin \phi_1)\ \Big[({T_{11}}^1 \cos \theta_8$$

$$+ {T_{21}}^1 \sin \psi_8) . {T_{33}}^9 + {T_{31}}^1 (\cos \theta_8 \cos \alpha_9$$

$$- \sin \theta_8 \cos \phi_8 \sin \alpha_9) \Big]$$

$$A\ (5,23) = -\ (C_{34} \cos \phi_1 + C_{35} \sin \phi_1)\ (T_{11}{}^1 T_{12}{}^8$$

$$+ T_{21}{}^1 T_{22}{}^8 + T_{31}{}^1 T_{32}{}^8) \sin \alpha_9$$

$$A\ (5,24) = (C_{34} \cos \phi_1 + C_{35} \sin \phi_1)\ (T_{11}{}^1 T_{11}{}^9$$

$$+ T_{21}{}^1 T_{21}{}^9 + T_{31}{}^1 T_{31}{}^9)$$

$$A\ (5,25) = (C_{32} \cos \phi_1 - C_{33} \sin \phi_1)\ (T_{11}{}^1 T_{21}{}^{10} - T_{21}{}^1 T_{11}{}^{10})$$

$$-\ (C_{34} \cos \phi_1 - C_{35} \sin \phi_1)\ (T_{11}{}^1 T_{23}{}^{11} - T_{21}{}^1 T_{13}{}^{11})$$

$$A\ (5,26) = (C_{32} \cos \phi_1 - C_{33} \sin \theta_1) \Big[(T_{11}{}^1 \cos \theta_{10}$$

$$+ T_{21}{}^1 \sin \psi_{10}) \sin \theta_{10} + T_{31}{}^1 \cos \theta_{10} \Big]$$

$$+ (C_{34} \cos \phi_1 - C_{35} \sin \phi_1) \Big[(T_{11}{}^1 \cos \psi_{10}$$

$$+ T_{21}{}^1 \sin \psi_{10})\ T_{33}{}^{11} + T_{31}{}^1 (\cos \theta_{10} \cos \alpha_{11}$$

$$- \sin \theta_{10} \cos \phi_{10} \sin \alpha_{11}) \Big]$$

$$A\ (5,27) = (C_{35} \sin \phi_1 - C_{34} \cos \phi_1)\ (T_{11}{}^1 T_{12}{}^{10}$$

$$+ T_{21}{}^1 T_{22}{}^{10} + T_{31}{}^1 T_{32}{}^{10}) \sin \alpha_{11}$$

$$A\ (5,28) = (C_{34} \cos \phi_1 - C_{35} \sin \phi_1)\ (T_{11}{}^1 T_{11}{}^{11}$$

$$+ T_{21}{}^1 T_{21}{}^{11} + T_{31}{}^1 T_{31}{}^{11})$$

$$A\ (6,6) = C_2 + C_6 + C_{37}$$

$$A\ (6,7) = C_{19}\ (T_{12}{}^1 T_{23}{}^3 - T_{22}{}^1 T_{13}{}^2)$$

$$A\ (6,8) = - C_{19} \cos\phi_2\ (T_{12}{}^1 T_{11}{}^2 + T_{22}{}^1 T_{21}{}^2 + T_{32}{}^1 T_{31}{}^2)$$

$$A\ (6,9) = C_{19}\ (T_{12}{}^1 T_{12}{}^2 + T_{22}{}^1 T_{22}{}^2 + T_{32}{}^1 T_{32}{}^2)$$

$$A\ (6,10) = C_{20}\ (T_{12}{}^1 T_{23}{}^3 - T_{22}{}^1 T_{13}{}^3)$$

$$+ C_{21}\ (T_{12}{}^1 T_{21}{}^3 - T_{22}{}^1 T_{11}{}^3)$$

$$A\ (6,11) = - C_{20} \cos\phi_3\ (T_{12}{}^1 T_{11}{}^3 + T_{22}{}^1\ T_{21}{}^3 + T_{32}{}^1 T_{31}{}^3)$$

$$+ C_{21} \Big[(T_{12}{}^1 \cos\psi_3 + T_{22}{}^1 \sin\psi_3) \sin\theta_3$$

$$+ T_{32}{}^1 \cos\theta_3\Big]$$

$$A\ (6,12) = C_{20}\ (T_{12}{}^1 T_{12}{}^3 + T_{22}{}^1 T_{22}{}^3 + T_{32}{}^1 T_{32}{}^3)$$

$$A\ (6,13) = C_{25}\ (T_{13}{}^4\ T_{22}{}^1 - T_{23}{}^4\ T_{12}{}^1)$$

$$- C_{28}\ (T_{11}{}^5 T_{22}{}^1 - T_{21}{}^5 T_{12}{}^1)$$

$$A\ (6,14) = C_{25} \cos\phi_4\ (T_{12}{}^1 T_{11}{}^4 + T_{22}{}^1 T_{21}{}^4 + T_{32}{}^1 T_{31}{}^4)$$

$$- C_{28} \Big[T_{31}{}^5\ (T_{12}{}^1 \cos\psi_4 + T_{22}{}^1 \sin\psi_4)$$

$$+ T_{32}{}^1\ (\sin\theta_4 \cos\phi_4 \cos\alpha_5 - \cos\theta_4 \sin\alpha_5)\Big]$$

$$A\ (6,15) = - (C_{25} + C_{28} \cos\alpha_5)\ (T_{12}{}^1 T_{12}{}^4$$

$$+ T_{22}{}^1 T_{22}{}^4 + T_{32}{}^1 T_{32}{}^4)$$

$$A\ (6,16) = -\ C_{28}\ (T_{12}{}^{1}T_{13}{}^{5} + T_{22}{}^{1}T_{23}{}^{5} + T_{32}{}^{1}T_{33}{}^{5})$$

$$A\ (6,17) = C_{25}\ (T_{13}{}^{6}T_{22}{}^{1} - T_{23}{}^{6}T_{12}{}^{1})$$

$$-\ C_{28}\ (T_{11}{}^{7}\ T_{22}{}^{1} - T_{21}{}^{7}\ T_{12}{}^{1})$$

$$A\ (6,18) = C_{25}\cos\phi_6\ (T_{12}{}^{1}T_{11}{}^{6} + T_{22}{}^{1}T_{21}{}^{6} + T_{32}{}^{1}T_{31}{}^{6})$$

$$-\ C_{28}\Big[T_{31}{}^{7}\ (T_{12}{}^{1}\cos\psi_6 + T_{22}1\sin\psi_6)$$

$$+\ T_{32}{}^{1}\ (\sin\theta_6\cos\phi_6\cos\alpha_7 - \cos\theta_6\sin\alpha_7)\Big]$$

$$A\ (6,19) = -\ (C_{25} + C_{28}\cos\alpha_7)\ (T_{12}{}^{1}T_{12}{}^{6}$$

$$+\ T_{22}{}^{1}T_{22}{}^{6} + T_{32}{}^{1}T_{32}{}^{6})$$

$$A\ (6,20) = -\ C_{28}\ (T_{12}{}^{1}T_{13}{}^{7} + T_{22}{}^{1}T_{23}{}^{7} + T_{32}{}^{1}T_{33}{}^{7})$$

$$A\ (6,21) = -\ C_{32}\ (T_{12}{}^{1}T_{21}{}^{8} - T_{22}{}^{1}T_{11}{}^{8})$$

$$+\ C_{34}\ (T_{12}{}^{1}T_{23}{}^{9} - T_{22}{}^{1}T_{13}{}^{9})$$

$$+\ C_{33}\ (T_{13}{}^{1}T_{21}{}^{8} - T_{23}{}^{1}T_{11}{}^{8})$$

$$-\ C_{35}\ (T_{13}{}^{1}T_{23}{}^{9} - T_{23}{}^{1}T_{13}{}^{9})$$

$$A\ (6,22) = -\ (C_{32}\sin\theta_8 + C_{34}T_{33}{}^{9})\ (T_{12}{}^{1}\cos\psi_8$$

$$+\ T_{22}{}^{1}\sin\psi_8)$$

$$- \Big[C_{32} \cos \theta_8 + C_{34} (\cos \theta_8 \cos \alpha_9$$

$$- \sin \theta_8 \cos \phi_8 \sin \alpha_9)\Big] T_{32}{}^1 + (C_{33} \sin \theta_8$$

$$+ C_{35} T_{33}{}^9) (T_{13}{}^1 \cos \psi_8 + T_{23}{}^1 \sin \psi_8)$$

$$+ \Big[C_{33} \cos \theta_8 + C_{35} (\cos \theta_8 \cos \alpha_9$$

$$- \sin \theta_8 \cos \phi_8 \sin \alpha_9)\Big] T_{33}{}^1$$

$$A\ (6,23) = \Big[C_{34} (T_{12}{}^1 T_{12}{}^8 + T_{22}{}^1 T_{22}{}^8 + T_{32}{}^1 T_{32}{}^8)$$

$$- C_{35} (T_{13}{}^1 T_{12}{}^8 + T_{23}{}^1 T_{22}{}^8 + T_{33}{}^1 T_{32}{}^8)\Big] \sin \alpha_9$$

$$A\ (6,24) = - C_{34} (T_{12}{}^1 T_{11}{}^9 + T_{22}{}^1 T_{21}{}^9 + T_{32}{}^1 T_{31}{}^9)$$

$$+ C_{35} (T_{13}{}^1 T_{11}{}^9 + T_{23}{}^1 T_{21}{}^9 + T_{33}{}^1 T_{31}{}^9)$$

$$A\ (6,25) = - C_{32} (T_{12}{}^1 T_{21}{}^{10} - T_{22}{}^1 T_{11}{}^{10}0$$

$$+ C_{34} (T_{12}{}^1 T_{23}{}^{11} - T_{22}{}^1 T_{13}{}^{11})$$

$$- C_{33} (T_{13}{}^1 T_{21}{}^{10} - T_{23}{}^1 T_{11}{}^{10})$$

$$+ C_{35} (T_{13}{}^1 T_{23}{}^{11} - T_{23}{}^1 T_{13}{}^{11})$$

$$A\ (6,26) = - C_{32} \Big[(T_{12}{}^1 \cos \psi_{10} + T_{22}{}^1 \sin \psi_{10}) \sin \theta_{10}$$

$$+ T_{32}{}^1 \cos \theta_{10}\Big]$$

$$- C_{34} \left[(T_{12}{}^{1} \cos \psi_{10} + T_{22}{}^{1} \sin \psi_{10}) \, T_{33}{}^{11} \right.$$

$$\left. + T_{32}{}^{1} (\cos \theta_{10} \cos \alpha_{11} - \sin \theta_{10} \cos \phi_{10} \sin \alpha_{11}) \right]$$

$$- C_{33} \left[(T_{13}{}^{1} \cos \psi_{10} + T_{23}{}^{1} \sin \psi_{10}) \sin \theta_{10} \right.$$

$$\left. + T_{33}{}^{1} \cos \theta_{10} \right] - C_{35} \left[(T_{13}{}^{1} \cos \psi_{10} \right.$$

$$+ T_{23}{}^{1} \sin \psi_{10}) \, T_{33}{}^{11} + T_{33}{}^{1} (\cos \theta_{10} \cos \alpha_{11}$$

$$\left. - \sin \theta_{10} \cos \phi_{10} \sin \alpha_{11}) \right]$$

$$A\,(6,27) = \left[C_{34} (T_{12}{}^{1} T_{11}{}^{10} + T_{22}{}^{1} T_{22}{}^{10} + T_{32}{}^{1} T_{32}{}^{10}) \right.$$

$$\left. + C_{35} (T_{13}{}^{1} T_{12}{}^{10} + T_{23}{}^{1} T_{22}{}^{10} + T_{33}{}^{1} T_{32}{}^{10}) \right] \sin \alpha_{11}$$

$$A\,(6,28) = - C_{34} (T_{12}{}^{1} T_{11}{}^{11} + T_{22}{}^{1} T_{21}{}^{11} + T_{32}{}^{1} T_{31}{}^{11})$$

$$- C_{35} (T_{13}{}^{1} T_{11}{}^{11} + T_{23}{}^{1} T_{21}{}^{11} + T_{33}{}^{1} T_{31}{}^{11})$$

$$A\,(7,7) = C_{3} (\sin^{2} \theta_{2} \cos^{2} \phi_{2} + \sin^{2} \phi_{2})$$

$$+ C_{5} (\sin^{2} \theta_{2} \sin^{2} \phi_{2} + \cos^{2} \phi_{2})$$

$$+ C_{39} \sin^{2} \theta_{2} + C_{40} \cos^{2} \phi_{2}$$

$$A\,(7,8) = (C_{3} - C_{5}) \, T_{32}{}^{2} \cos \phi_{2}$$

$$A\ (7,9) = (C_3 + C_5 + C_{39})\ T_{31}^{\ 2}$$

$$A\ (7,10) = C_{22}\ (T_{13}^{\ 2} T_{13}^{\ 3} + T_{23}^{\ 2}\ T_{23}^{\ 3})$$

$$+ C_{23}\ (T_{13}^{\ 2}\ T_{11}^{\ 3} + T_{23}^{\ 2} T_{21}^{\ 3})$$

$$A\ (7,11) = (C_{22} T_{33}^{\ 3} + C_{23} T_{31}^{\ 3})\ (T_{13}^{\ 2} \sin \psi_3 - T_{23}^{\ 3} \cos \psi_3)$$

$$A\ (7,12) = C_{22}\ (T_{12}^{\ 3} T_{23}^{\ 2} - T_{22}^{\ 3} T_{13}^{\ 2})$$

$$A\ (7,13) = -\ C_{26}\ (T_{13}^{\ 2} T_{13}^{\ 4} + T_{23}^{\ 2} T_{23}^{\ 4})$$

$$+ C_{27}\ (T_{12}^{\ 2} T_{13}^{\ 4} + T_{22}^{\ 2} T_{23}^{\ 4})$$

$$+ C_{29}\ (T_{13}^{\ 2} T_{11}^{\ 5} + T_{23}^{\ 2} T_{21}^{\ 5})$$

$$-\ C_{30}\ (T_{12}^{\ 2} T_{11}^{\ 5} + T_{22}^{\ 2} T_{21}^{\ 5})$$

$$A\ (7,14) = (C_{26} T_{33}^{\ 4} - C_{29} T_{31}^{\ 5})\ (T_{23}^{\ 2} \cos \psi_4 - T_{13}^{\ 2} \sin \psi_4)$$

$$-\ (C_{27} T_{33}^{\ 9} - C_{30} T_{31}^{\ 5})\ (T_{22}^{\ 2} \cos \psi_4 - T_{12}^{\ 2} \sin \psi_4)$$

$$A\ (7,15) = (C_{26} + C_{29} \cos \alpha_5)\ (T_{13}^{\ 2} T_{22}^{\ 4} - T_{23}^{\ 2} T_{12}^{\ 4})$$

$$-\ (C_{27} + C_{30} \cos \alpha_5)\ (T_{12}^{\ 2} T_{22}^{\ 4} - T_{22}^{\ 2} T_{12}^{\ 4})$$

$$A\ (7,16) = C_{29}\ (T_{13}^{\ 2} T_{23}^{\ 5} - T_{23}^{\ 2} T_{13}^{\ 5})$$

$$-\ C_{30}\ (T_{12}^{\ 2} T_{23}^{\ 5} - T_{22}^{\ 2} T_{13}^{\ 5})$$

$$A\ (7,17) = -\ C_{26}\ (T_{13}^{\ 2}T_{13}^{\ 6} + T_{23}^{\ 2}T_{23}^{\ 6})$$

$$-\ C_{27}\ (T_{12}^{\ 2}T_{13}^{\ 6} + T_{22}^{\ 2}T_{23}^{\ 6})$$

$$+\ C_{29}\ (T_{13}^{\ 2}T_{11}^{\ 7} + T_{23}^{\ 2}T_{21}^{\ 7})$$

$$+\ C_{30}\ (T_{12}^{\ 2}T_{11}^{\ 7} + T_{22}^{\ 2}T_{21}^{\ 7})$$

$$A\ (7,18) = (C_{26}T_{33}^{\ 6} - C_{29}T_{31}^{\ 7})\ (T_{23}^{\ 2} \cos \psi_6 - T_{13}^{\ 2} \sin \psi_6)$$

$$+\ (C_{27}T_{33}^{\ 6} - C_{30}T_{31}^{\ 7})\ (T_{22}^{\ 2} \cos \psi_6 - T_{12}^{\ 2} \sin \psi_6)$$

$$A\ (7,19) = (C_{26} + C_{29} \cos \alpha_7)\ (T_{13}^{\ 2}T_{22}^{\ 6} - T_{23}^{\ 2}T_{12}^{\ 6})$$

$$+\ (C_{27} + C_{30} \cos\ {}_{7})\ (T_{12}^{\ 2}T_{22}^{\ 6} - T_{22}^{\ 2}T_{12}^{\ 6})$$

$$A\ (7,20) = C_{29}\ (T_{13}^{\ 2}T_{23}^{\ 7} - T_{23}^{\ 2}T_{13}^{\ 7})$$

$$+\ C_{30}\ (T_{12}^{\ 2}T_{23}^{\ 7} - T_{22}^{\ 2}T_{13}^{\ 7})$$

$$A\ (8,8) = C_3 \cos^2 \phi_2 + C_5 \sin^2 \phi_2 + C_{40}$$

$$A\ (8,10) = C_{22} \Big[(T_{13}^{\ 3}T_{21}^{\ 2} - T_{23}^{\ 2}T_{11}^{\ 2})$$

$$+\ C_{23}\ (T_{11}^{\ 3}T_{21}^{\ 2} - T_{21}^{\ 3}T_{11}^{\ 2})\Big] \cos \phi_2$$

$$A\ (8,11) = \Big\{ C_{22} \cos \phi_3\ (T_{11}^{\ 2}T_{11}^{\ 3} + T_{21}^{\ 2}T_{21}^{\ 3} + T_{31}^{\ 2}T_{31}^{\ 3})$$
$$-\ C_{23} \Big[(T_{11}^{\ 2} \cos \psi_3 + T_{21}^{\ 2} \sin \psi_3) \sin \theta_3$$
$$+\ T_{31}^{\ 2} \cos \theta_3 \Big] \Big\} \cos \phi_2$$

$$A\ (8,12) = -\ C_{22} \cos \phi_2\ (T_{11}{}^2 T_{12}{}^3 + T_{21}{}^2 T_{22}{}^3 + T_{31}{}^2 T_{32}{}^3)$$

$$A\ (8,13) = (C_{26} \cos \phi_2 - C_{27} \sin \phi_2)\ (T_{11}{}^2 T_{23}{}^4 - T_{21}{}^2 T_{13}{}^4)$$

$$-\ (C_{29} \cos \phi_2 - C_{30} \sin \phi_2)\ (T_{11}{}^2 T_{21}{}^5 - T_{21}{}^2 T_{11}{}^5)$$

$$A\ (8,14) = -\ (C_{26} \cos \phi_2 - C_{27} \sin \phi_2)\ (T_{11}{}^2 T_{11}{}^4 + T_{21}{}^2 T_{21}{}^4$$

$$+ T_{31}{}^2 T_{31}{}^4) \cos \phi_4$$

$$+ (C_{29} \cos \phi_2 - C_{30} \sin \phi_2) \Big[(T_{11}{}^2 \cos \psi_4$$

$$+ T_{21}{}^2 \sin \psi_4)\ T_{31}{}^5 + T_{31}{}^2 (\sin \theta_4 \cos \phi_4 \cos \alpha_5$$

$$- \cos \theta_4 \sin \alpha_5) \Big]$$

$$A\ (8,15) = \Big[(C_{26} + C_{29} \cos \alpha_5) \cos \phi_2 - (C_{27}$$

$$+ C_{30} \cos \alpha_5) \sin \phi_2 \Big]\ (T_{11}{}^2 T_{12}{}^4 + T_{21}{}^2 T_{22}{}^4$$

$$+ T_{31}{}^2 T_{32}{}^4)$$

$$A\ (8,16) = (C_{29} \cos \phi_2 - C_{30} \sin \phi_2)\ (T_{11}{}^2 T_{13}{}^5$$

$$+ T_{21}{}^2 T_{23}{}^5 + T_{31}{}^2 T_{33}{}^5)$$

$$A\,(8,17) = (C_{26}\cos\phi_2 + C_{27}\sin\phi_2)\,({T_{11}}^{2}{T_{23}}^{6} - {T_{21}}^{2}{T_{13}}^{6})$$

$$- (C_{29}\cos\phi_2 + C_{30}\sin\phi_2)\,({T_{11}}^{2}{T_{21}}^{7} - {T_{21}}^{2}{T_{11}}^{7})$$

$$A\,(8,18) = -(C_{26}\cos\phi_2 + C_{27}\sin\phi_2)\,({T_{11}}^{2}{T_{11}}^{6} + {T_{21}}^{2}{T_{21}}^{6}$$

$$+ {T_{31}}^{2}{T_{31}}^{6})\ \cos\phi_6 + (C_{29}\cos\phi_2$$

$$+ C_{30}\sin\phi_2)\left[({T_{11}}^{2}\cos\psi_6 + {T_{21}}^{2}\sin\psi_6)\,{T_{31}}^{7}\right.$$

$$\left. + {T_{31}}^{2}\,(\sin\theta_6\cos\phi_6\cos\alpha_7 - \cos\theta_6\sin\alpha_7)\right]$$

$$A\,(8,19) = \left[(C_{26} + C_{29}\cos\alpha_7)\cos\phi_2 + (C_{27}\right.$$

$$\left. + C_{30}\cos\alpha_7)\sin\phi_2\right]\,({T_{11}}^{2}{T_{12}}^{6} + {T_{21}}^{2}{T_{22}}^{6}$$

$$+ {T_{31}}^{2}{T_{32}}^{6})$$

$$A\,(8,20) = (C_{29}\cos\phi_2 + C_{30}\sin\phi_2)\,({T_{11}}^{2}{T_{13}}^{7} + {T_{21}}^{2}{T_{23}}^{7}$$

$$+ {T_{31}}^{2}{T_{33}}^{7})$$

$$A\,(9,9) = C_3 + C_5 + C_{39}$$

$$A\,(9,10) = C_{22}\,({T_{12}}^{2}{T_{23}}^{3} - {T_{22}}^{2}{T_{13}}^{3})$$

$$+ C_{23}\,({T_{12}}^{2}{T_{21}}^{3} - {T_{22}}^{2}{T_{11}}^{3})$$

$$A\ (9,11) = -\ C_{22} \cos \phi_3 \ (T_{12}^{2} T_{11}^{3} + T_{22}^{2} T_{21}^{3} + T_{32}^{2} T_{31}^{3})$$

$$+\ C_{23} \left[(T_{12}^{2} \cos \psi_3 + T_{22}^{2} \sin \psi_3) \sin \theta_3 \right.$$

$$\left. +\ T_{32}^{2} \cos \theta_3 \right]$$

$$A\ (9,12) = C_{22} \ (T_{12}^{2} T_{12}^{3} + T_{22}^{2} T_{22}^{3} + T_{32}^{2} T_{32}^{3})$$

$$A\ (9,13) = -\ C_{26} \ (T_{12}^{2} T_{23}^{4} - T_{22}^{2} T_{13}^{4})$$

$$-\ C_{27} \ (T_{13}^{2} T_{23}^{4} - T_{23}^{2} T_{13}^{4})$$

$$+\ C_{29} \ (T_{12}^{2} T_{21}^{5} - T_{22}^{2} T_{11}^{5})$$

$$+\ C_{30} \ (T_{13}^{2} T_{21}^{5} - T_{23}^{2} T_{11}^{5})$$

$$A\ (9,14) = \left[C_{26} \ (T_{12}^{2} T_{11}^{4} + T_{22}^{2} T_{21}^{4} + T_{32}^{2} T_{31}^{4}) \right.$$

$$\left. +\ C_{27} \ (T_{13}^{2} T_{11}^{4} + T_{23}^{2} T_{21}^{4} + T_{33}^{2} T_{31}^{4}) \right] \cos \phi_4$$

$$-\ C_{29} \left[(T_{12}^{2} \cos \psi_4 + T_{22}^{2} \sin \psi_4) \ T_{31}^{5} \right.$$

$$\left. +\ T_{32}^{2} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5) \right]$$

$$-\ C_{30} \left[(T_{13}^{2} \cos \psi_4 + T_{23}^{2} \sin \psi_4) \ T_{31}^{5} \right.$$

$$\left. +\ T_{33}^{2} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5) \right]$$

$$A\,(9,15) = -\,(C_{26} + C_{29}\cos\alpha_5)\,(T_{12}{}^{2}T_{12}{}^{4} + T_{22}{}^{2}T_{22}{}^{4}$$

$$+\,T_{32}{}^{2}T_{32}{}^{4})$$

$$-\,(C_{27} + C_{30}\cos\alpha_5)\,(T_{13}{}^{2}T_{12}{}^{4} + T_{23}{}^{2}T_{22}{}^{4}$$

$$+\,T_{33}{}^{2}T_{32}{}^{4})$$

$$A\,(9,16) = -\,C_{29}\,(T_{12}{}^{2}T_{13}{}^{5} + T_{22}{}^{2}T_{23}{}^{5} + T_{32}{}^{2}T_{33}{}^{5})$$

$$-\,C_{30}(T_{13}{}^{2}T_{13}{}^{5} + T_{23}{}^{2}T_{23}{}^{5} + T_{33}{}^{2}T_{33}{}^{5})$$

$$A\,(9,17) = -\,C_{26}\,(T_{12}{}^{2}T_{23}{}^{6} - T_{22}{}^{2}T_{13}{}^{6})$$

$$+\,C_{27}\,(T_{13}{}^{2}T^{23}{}_{6} - T_{23}{}^{3}\,T_{13}{}^{6})$$

$$+\,C_{29}\,(T_{12}{}^{2}T_{21}{}^{7} - T_{22}{}^{2}T_{11}{}^{7})$$

$$-\,C_{30}\,(T_{13}{}^{2}T_{21}{}^{7} - T_{23}{}^{2}T_{11}{}^{7})$$

$$A\,(9,18) = \Big[C_{26}\,(T_{12}{}^{2}T_{11}{}^{6} + T_{22}{}^{2}T_{21}{}^{6} + T_{32}{}^{2}T_{31}{}^{6})$$

$$-\,C_{27}\,(T_{12}{}^{2}T_{11}{}^{6} + T_{22}{}^{2}T_{21}{}^{6} + T_{32}{}^{2}T_{31}{}^{6})\Big]\cos\phi_6$$

$$-\,C_{29}\,\Big[(T_{12}{}^{2}\cos\psi_6 + T_{22}{}^{2}\sin\psi_6)\,T_{31}{}^{7}$$

$$+\,T_{32}{}^{2}\,(\sin\theta_6\cos\phi_6\cos\alpha_7 - \cos\theta_6\sin\alpha_7)\Big]$$

$$+\,C_{30}\,\Big[(T_{13}{}^{2}\cos\psi_6 + T_{22}{}^{2}\sin\psi_6)\,T_{31}{}^{7}$$

$$+\,T_{33}{}^{2}\,(\sin\theta_6\cos\phi_6\cos\alpha_7 - \cos\theta_6\sin\alpha_7)\Big]$$

$$A\ (9,19) = -\ (C_{26} + C_{29} \cos \alpha_7)\ (T_{12}{}^{2}T_{12}{}^{6} + T_{22}{}^{2}T_{22}{}^{6}$$

$$+ T_{32}{}^{2}T_{32}{}^{6}) + (C_{27} + C_{30} \cos \alpha_7)\ (T_{13}{}^{2}T_{12}{}^{6}$$

$$+ T_{23}{}^{2}T_{22}{}^{6} + T_{33}{}^{2}T_{33}{}^{6})$$

$$A\ (9,20) = -\ C_{29}\ (T_{12}{}^{2}T_{13}{}^{7} + T_{22}{}^{2}T_{23}{}^{7} + T_{32}{}^{2}T_{33}{}^{7})$$

$$+ C_{30}\ (T_{13}{}^{2}T_{13}{}^{7} + T_{23}{}^{2}T_{23}{}^{7} + T_{33}{}^{2}T_{33}{}^{7})$$

$$A\ (10,10) = C_{13}\rho_3\ (\sin^2 \theta_3 \cos^2 \phi_3 + \sin^2 \phi_3)$$

$$+ C_{41}\ (\sin^2 \theta_3 + \cos^2 \theta_3 \sin^2 \phi_3)$$

$$+ C_{42} \cos^2 \theta_3 \cos^2 \phi_3 + C_4 \cos^2 \theta_3$$

$$+ 2\ C_{24}T_{33}{}^{3} \sin \theta_3$$

$$A\ (10,11) = \left[(C_{13}\rho_3 + C_{41} - C_{42})\ T_{33}{}^{3} + C_{24}T_{31}{}^{3}\right] \sin \phi_3$$

$$A\ (10,12) = -\ (C_{13}\rho_3 + C_{41}) \sin \theta_3 - C_{24}T_{33}{}^{3}$$

$$= (C_{13}\rho_3 + C_{41})\ T_{31}{}^{3} - C_{24}T_{33}{}^{3}$$

$$A\ (11,11) = (C_{13}\rho_3 + C_{41}) \cos^2 \phi_3 + C_{42} \sin^2 \phi_3 + C_4$$

$$A\ (11,12) = C_{24} \sin \phi_3$$

$$A\ (12,12) = C_{13}\rho_3 + C_{41}$$

$$A\ (13,13) = (C_7 + 2\ C_{31} \cos\alpha_5)\ (\sin^2\theta_4 \cos^2\phi_4 + \sin^2\phi_4)$$

$$+ C_8 \left[(\cos\theta_4 \sin\alpha_5 - \sin\theta_4 \cos\phi_4 \cos\alpha_5)^2\right.$$

$$\left. + (\sin\phi_4 \cos\alpha_5)^2\right] + C_{43} \left[({T_{31}}^4)^2 + ({T_{32}}^4)^2\right]$$

$$+ C_{44}\ ({T_{33}}^4)^2 + C_{46}\ ({T_{33}}^5)^2 + C_{45}\ ({T_{31}}^5)^2$$

$$+ 2\ C_{31}{T_{31}}^4{T_{33}}^4 \sin\alpha_5 + C_{46}\ ({T_{32}}^4)^2$$

$$A\ (13,14) = \left[C_7 + C_{43} - C_{44} + 2\ C_{31} \cos\alpha_5\right.$$

$$\left. + (C_8 - C_{45} + C_{46}) \cos^2\alpha_5\right] \cos\theta_4 \sin\phi_4 \cos\phi_4$$

$$+ \left[(C_8 - C_{45} + C_{46}) \cos\alpha_5\right.$$

$$\left. + C_{31}\right] \sin\theta_4 \sin\phi_4 \sin\alpha_5$$

$$- C_8 \sin\psi_4 \cos\psi_4 \sin\theta_4 \cos\theta_4 \sin^2\phi_4 \cos^2\alpha_5$$

$$A\ (13,15) = (C_7 + C_{31} \cos\alpha_5 + C_{43})\ {T_{31}}^4$$

$$+ (C_8 + C_{46})\ {T_{33}}^5 \cos\alpha_5 + C_{31}{T_{33}}^5$$

$$+ C_{45}{T_{31}}^5 \sin\alpha_5$$

$$A\ (13,16) = (C_{46} - C_8 - C_{31} \cos\alpha_5)\ {T_{32}}^4$$

$$A\ (14,14) = (C_7 + C_{43} + C_{46} + C_8 \cos^2 \alpha_5$$

$$+ 2\ C_{31} \cos \alpha_5) \cos^2 \phi_4$$

$$+ (C_{44} + C_{45} \cos^2 \alpha_5 + C_{46} \sin^2 \alpha_5) \sin^2 \phi_4$$

$$+ C_8 \sin^2 \alpha_5$$

$$A\ (14,15) = - \left[(C_8 - C_{45} + C_{46}) \cos \alpha_7 + C_{31}\right] \sin \phi_6 \sin \alpha_7$$

$$A\ (14,16) = (C_{46} - C_8 - C_{31} \cos \alpha_5) \cos \phi_4$$

$$A\ (15,15) = C_7 + C_{43} + 2\ C_{31} \cos \alpha_5 + C_{45} \sin^2 \alpha_5$$

$$+ (C_8 + C_{46}) \cos^2 \alpha_5$$

$$A\ (16,16) = C_8 + C_{46}$$

$$A\ (17,17) = \left[(C_7 + 2\ C_{31} \cos \alpha_7 + C_8 \cos^2 \alpha_7) \cos^2 \phi_6\right.$$

$$\left. + C_{43} + C_{45} \sin^2 \alpha_7 + C_{46} \cos^2 \alpha_7\right] \sin^2 \theta_6$$

$$+ (C_{44} + C_{46} \cos^2 \alpha_7 + C_{46} \sin^2 \alpha_7)\ (T_{33}{}^6)^2$$

$$+ (C_{43} + C_{46})\ (T_{32}{}^6)^2$$

$$+ (C_7 + C_8 \cos^2 \alpha_7 + 2\ C_{31} \cos \alpha_7) \sin^2 \phi_6$$

$$- 2 \left[(C_8 - C_{45} + C_{46}) \cos \alpha_7\right.$$

$$\left. + C_{31}\right] \sin \theta_6 T_{33}{}^6 \sin \alpha_7 + C_8 \cos^2 \theta_6 \sin^2 \alpha_7$$

$$A\ (17,18) = \left[C_7 + C_{43} - C_{44} + 2\ C_{31} \cos \alpha_7\right.$$

$$\left. + (C_8 - C_{45} + C_{46}) \cos^2 \alpha_7\right] T_{32}{}^6 \cos \phi_6$$

$$+ \left[(C_8 - C_{45} + C_{46}) \cos \alpha_7\right.$$

$$\left. + C_{31}\right] \sin \theta_6 \sin \phi_6 \sin \alpha_7$$

$$- C_8 \sin \psi_6 \cos \psi_6 \sin \theta_6 \cos \theta_6 \sin^2 \phi_6 \cos^2 \alpha_7$$

$$A\ (17,19) = (C_7 + C_{43} + C_{31} \cos \alpha_7)\ T_{31}{}^6$$

$$+ \left[(C_8 + C_{46}) \cos \alpha_7 + C_{31}\right] T_{33}{}^7 + C_{45} T_{31}{}^7 \sin \alpha_7$$

$$A\ (17,20) = (C_{46} - C_8 - C_{31} \cos \alpha_7)\ T_{32}{}^6$$

$$A\ (18,18) = (C_7 + C_{43} + 2\ C_{31} \cos \alpha_7 + C_8 \cos^2 \alpha_7$$

$$+ C_{46}) \cos^2 \phi_6 + (C_{44} + C_{45} \cos^2 \alpha_7$$

$$+ C_{46} \sin^2 \alpha_7) \sin^2 \phi_6 + C_8 \sin^2 \alpha_7$$

$$A\ (18,19) = - \left[(C_8 - C_{45} + C_{46}) \cos \alpha_7 + C_{31}\right] \sin \phi_6 \sin \alpha_7$$

$$A\ (18,20) = (C_{46} - C_8 - C_{31} \cos \alpha_7) \cos \phi_6$$

$$A\ (19,19) = C_7 + C_{43} + 2\, C_{31} \cos \alpha_7$$

$$+ (C_8 + C_{46}) \cos^2 \alpha_7 + C_{45} \sin^2 \alpha_7$$

$$A\ (20,20) = C_8 + C_{46}$$

$$A\ (21,21) = \Big[C_9 + C_{48} + (C_{49} \cos^2 \phi_8 + C_{10}) \cos^2 \alpha_9$$

$$+ C_{49} \sin^2 \phi_8 + 2\, C_{36} \cos \alpha_9$$

$$+ C_{50} \cos^2 \phi_8 \sin^2 \alpha_9 \Big] \cos^2 \theta_8$$

$$+ \Big[C_{47} + (C_{10} \cos^2 \phi_8 + C_{49}) \sin^2 \alpha_9$$

$$+ C_{50} \cos^2 \alpha_9 \Big] \sin^2 \theta_8 + C_{10} \sin^2 \phi_8 \sin^2 \alpha_9$$

$$- 2 \Big[(C_{10} - C_{49} - C_{50}) \cos \alpha_9$$

$$+ C_{36} \Big] \sin \theta_8\, T_{33}{}^{8} \sin \alpha_9$$

$$A\ (21,22) = (C_{10} + C_{49} - C_{50})\, T_{33}{}^{9}$$

$$+ C_{36} \sin \theta_8 \sin \phi_8 \sin \alpha_9$$

$$A\ (21,23) = \Big[(C_{10} + C_{49} - C_{50}) \cos \alpha_9 + C_{36} \Big] T_{33}{}^{8} \sin \alpha_9$$

$$- \Big[(C_{10} + C_{49}) \sin^2 \alpha_9 + C_{50} \cos^2 \alpha_9 + C_{47} \Big] \sin \theta_8$$

$$A\ (21,24) = (C_{10} + C_{49} + C_{36} \cos \alpha_9)\, T_{32}{}^{8}$$

$$A\ (22,22) = C_9 + C_{48} + 2\ C_{36} \cos \alpha_9$$

$$+ (C_{10} + C_{49} \sin^2 \phi_8) \cos^2 \alpha_9 + C_{49} \cos^2 \phi_8$$

$$+ (C_{10} \cos^2 \phi_8 + C_{50} \sin^2 \phi_8) \sin^2 \alpha_9$$

$$A\ (22,23) = - \left[(C_{10} + C_{49} - C_{50}) \cos \alpha_9 + C_{36}\right] \sin \phi_8 \sin \alpha_9$$

$$A\ (22,24) = (C_{10} + C_{49} + C_{36} \cos \alpha_9) \cos \phi_8$$

$$A\ (23,23) = (C_{10} + C_{49}) \sin^2 \alpha_9 + C_{50} \cos^2 \alpha_9 + C_{47}$$

$$A\ (24,24) = C_{10} + C_{49}$$

$$A\ (25,25) = \Big[C_9 + C_{48} + (C_{10} + C_{49} \cos^2 \phi_{10}) \cos^2 \alpha_{11}$$

$$+ 2\ C_{36} \cos \alpha_{11} + C_{49} \sin^2 \phi_{10}$$

$$+ C_{50} \cos^2 \phi_{10} \sin^2 \alpha_{11}\Big] \cos^2 \theta_{10}$$

$$+ \Big[(C_{10} \cos^2 \phi_{10} + C_{49}) \sin^2 \alpha_{11} + C_{47}$$

$$+ C_{50} \cos^2 \alpha_{11}\Big] \sin^2 \theta_{10} + C_{10} \sin^2 \phi_{10} \sin^2 \alpha_{11}$$

$$+ 2 \Big[(C_{10} + C_{49} - C_{50}) \cos \alpha_{11}$$

$$+ C_{36}\Big]\ T_{31}{}^{10} T_{33}{}^{10} \sin \alpha_{11}$$

$$A\ (25,26) = \Big[(C_{10} - C_{49} - C_{50})\ T_{33}{}^{1}$$

$$+ C_{36} \sin \theta_{10}\Big] \sin \phi_{10} \sin \alpha_{11}$$

$$A\ (25,27) = \Big[(C_{10} + C_{49} - C_{50}) \cos \alpha_{11} + C_{36}\Big]\ T_{33}{}^{10} \sin \alpha_{11}$$

$$+ \Big[(C_{10} + C_{49}) \sin^2 \alpha_{11} + C_{50} \cos^2 \alpha_{11} + C_{47}\Big]\ T_{31}{}^{10}$$

$$A\ (25,28) = (C_{10} + C_{49} + C_{36} \cos \alpha_{11})\ T_{32}{}^{10}$$

$$A\ (26,26) = C_9 + C_{48} + 2\ C_{36} \cos \alpha_{11}$$

$$+ (C_{10} \cos^2 \phi_{10} + C_{50} \sin^2 \phi_{10}) \sin^2 \alpha_{11}$$

$$+ (C_{10} + C_{49} \sin^2 \phi_{10}) \cos^2 \alpha_{11} + C_{49} \cos^2 \phi_{10}$$

$$A\ (26,27) = - \Big[(C_{10} + C_{49} - C_{50}) \cos \alpha_{11} + C_{36}\Big] \sin \phi_{10} \sin \alpha_{11}$$

$$A\ (26,28) = (C_{10} + C_{49} + C_{36} \cos \alpha_{11}) \cos \phi_{10}$$

$$A\ (27,27) = (C_{10} + C_{49}) \sin^2 \alpha_{11} + C_{50} \cos^2 \alpha_{11} + C_{47}$$

$$A\ (28,28) = C_{10} + C_{49}$$

$$B\ (1) = C_{11}D_{13}{}^{1} + C_{12}D_{13}{}^{2} + C_{13}D_{13}{}^{3} + C_{14}D_{11}{}^{3}$$

$$- C_{15}\ (D_{13}{}^{4} + D_{13}{}^{6}) + C_{16}\ (D_{11}{}^{5} + D_{11}{}^{7})$$

$$+ C_{17}\ (D_{11}{}^{8} + D_{11}{}^{10}) - C_{18}\ (D_{13}{}^{9} + D_{13}{}^{1})$$

$$B\ (2) = C_{11}D_{23}{}^{1} + C_{12}D_{23}{}^{2} + C_{13}D_{23}{}^{3} + C_{14}D_{21}{}^{3}$$

$$- C_{15}\ (D_{23}{}^{4} + D_{23}{}^{6}) + C_{16}\ (D_{21}{}^{5} + D_{21}{}^{7})$$

$$+ C_{17}\ (D_{21}{}^{8} + D_{21}{}^{10}) - C_{18}\ (D_{23}{}^{9} + D_{23}{}^{11})$$

$$B\ (3) = C_{11}D_{33}{}^{1} + C_{12}D_{33}{}^{2} + C_{13}D_{33}{}^{3} + C_{14}D_{31}{}^{3}$$

$$- C_{15}\ (D_{33}{}^{4} + D_{33}{}^{6}) + C_{16}\ (D_{31}{}^{5} + D_{31}{}^{7})$$

$$+ C_{17}\ (D_{31}{}^{8} + D_{31}{}^{10}) - C_{18}\ (D_{33}{}^{9} + D_{33}{}^{11})$$

$$B\ (4) = -\ 2\dot{\psi}_1\dot{\theta}_1 \Big[C_2 \cos^2 \phi_1 + (C_6 + C_{37}$$

$$- C_{38}) \sin^2 \phi_1\Big] \sin \theta_1 \cos \theta_1$$

$$+ 2\dot{\psi}_1\dot{\phi}_1\ T_{32}{}^{1}T_{33}{}^{1}\ (C_6 - C_2 + C_{37} - C_{38})$$

$$- \dot{\theta}_1^{\,2} \, (C_6 - C_2 + C_{37} - C_{38}) \sin \theta_1 \sin \phi_1 \cos \phi_1$$

$$+ \dot{\theta}_1 \dot{\theta}_1 \Big[(2\, C_2 + C_{38}) \sin^2 \phi_1 + (2\, C_6 + 2\, C_{37}$$

$$- C_{37}) \cos^2 \phi_1 \Big] \cos \theta_1$$

$$+ C_{19} \, (T_{13} \, {}^{1}D_{23}{}^{2} - T_{23} \, {}^{1}D_{13}{}^{2})$$

$$+ C_{20} \, (T_{13} \, {}^{1}D_{23}{}^{3} - T_{23} \, {}^{1}D_{13}{}^{3})$$

$$+ C_{21} \, (T_{13} \, {}^{1}D_{21}{}^{3} - T_{23} \, {}^{1}D_{11}{}^{3})$$

$$- C_{25} \, (T_{13} \, {}^{1}D_{23}{}^{4} - T_{23} \, {}^{1}D_{13}{}^{4})$$

$$- C_{25} \, (T_{13} \, {}^{1}D_{23}{}^{6} - T_{23} \, {}^{1}D_{13}{}^{6})$$

$$+ C_{28} \, (T_{13} \, {}^{1}D_{21}{}^{5} - T_{23} \, {}^{1}D_{11}{}^{5})$$

$$+ C_{28} \, (T_{13} \, {}^{1}D_{21}{}^{7} - T_{23} \, {}^{1}D_{11}{}^{7})$$

$$- C_{32} \, (T_{13} \, {}^{1}D_{21}{}^{8} - T_{23} \, {}^{1}D_{11}{}^{8})$$

$$- C_{32} \, (T_{13} \, {}^{1}D_{21}{}^{10} - T_{23} \, {}^{1}D_{11}{}^{10})$$

$$- C_{33} \, (T_{12} \, {}^{1}D_{21}{}^{8} - T_{22} \, {}^{1}D_{11}{}^{8})$$

$$+ C_{33} \, (T_{12} \, {}^{1}D_{21}{}^{10} - T_{22} \, {}^{1}D_{11}{}^{10})$$

$$+ (C_{34}T_{13}{}^{1} + C_{35}T_{12}{}^{1})\ D_{23}{}^{9}$$

$$- (C_{34}T_{23}{}^{1} + C_{35}T_{22}{}^{1})\ D_{13}{}^{9}$$

$$+ (C_{34}T_{13}{}^{1} - C_{35}T_{12}{}^{1})\ D_{23}{}^{11}$$

$$- (C_{34}T_{23}{}^{1} - C_{35}T_{22}{}^{1})\ D_{13}{}^{11}$$

$$B\ (5) = \dot{\psi}_1{}^2 \Big[C_2 \cos^2 \phi_1 + (C_6 + C_{37}$$

$$- C_{38}) \sin^2 \phi_1\Big] \sin \theta_1 \cos \theta_1$$

$$- \dot{\psi}_1 \dot{\phi}_1 \cos \theta_1 \Big[(2\ C_2 + C_{38}) \cos^2 \phi_1 + (2\ C_6 + 2\ C_{37}$$

$$- C_{38}) \sin^2 \phi_1\Big]$$

$$+ 2\dot{\theta}_1 \dot{\phi}_1 \sin \phi_1 \cos \phi_1 (C_2 - C_6 - C_{37} + C_{38})$$

$$+ \Big[C_{19}\ (D_{13}{}^{2}T_{11}{}^{1} + D_{23}{}^{2}T_{21}{}^{1} + D_{33}{}^{2}T_{31}{}^{1})$$

$$+ C_{20}\ (D_{13}{}^{3}T_{11}{}^{1} + D_{23}{}^{3}T_{21}{}^{1} + D_{33}{}^{3}T_{31}{}^{1})$$

$$+ C_{21}\ (D_{11}{}^{3}T_{11}{}^{1} + D_{21}{}^{3}T_{21}{}^{1} + D_{31}{}^{3}T_{31}{}^{1})$$

$$- C_{25}\ (D_{13}{}^{4}T_{11}{}^{1} + D_{23}{}^{4}T_{21}{}^{1} + D_{33}{}^{4}T_{31}{}^{1}$$

$$+ D_{13}{}^{6}T_{11}{}^{1} + D_{23}{}^{6}T_{21}{}^{1} + D_{33}{}^{6}T_{31}{}^{1})$$

$$+ C_{28} (\{D_{11}^{5} + D_{11}^{7}\} T_{11}^{1}$$

$$+ \{D_{21}^{5} + D_{21}^{7}\} T_{21}^{1} + \{D_{31}^{5} + D_{31}^{7}\} T_{31}^{1})$$

$$- C_{32} (\{D_{11}^{8} + D_{11}^{10}\} T_{11}^{1}$$

$$+ \{D_{21}^{8} + D_{21}^{10}\} T_{21}^{1} + \{D_{31}^{8} + D_{31}^{1}\} T_{31}^{1})$$

$$+ C_{34} (\{D_{13}^{9} + D_{13}^{11}\} T_{11}^{1} + \{D_{23}^{9} + D_{23}^{11}\} T_{21}^{1}$$

$$+ \{D_{33}^{9} + D_{33}^{1}\} T_{31}^{1})\Big] \cos \phi_1$$

$$- \Big[C_{33} (\{D_{11}^{8} - D_{11}^{10}\} T_{11}^{1} + \{D_{21}^{8} - D_{21}^{10}\} T_{21}^{1}$$

$$+ \{D_{31}^{8} - D_{31}^{10}\} T_{31}^{1})$$

$$- C_{35} (\{D_{13}^{9} - D_{13}^{11}\} T_{11}^{1} + \{D_{23}^{9} - D_{23}^{1}\} T_{21}^{1}$$

$$+ \{D_{33}^{9} - D_{33}^{11}\} T_{31}^{1})\Big] \sin \phi_1$$

$$B\ (6) = (\dot{\psi}_1^{2} \sin^2 \theta_1 - \dot{\theta}_1^{2}) (C_2 - C_6 - C_{37} + D_{38}) \sin \phi_1 \cos \phi_1$$

$$+ \dot{\psi}_1 \dot{\theta}_1 \cos \theta_1 \Big[(2\ C_2 + C_{38}) \cos^2 \phi_1 + (2\ C_6 + 2\ C_{37}$$

$$- C_{38}) \sin^2 \phi_1\Big]$$

$$- C_{19} (D_{13}^{2} T_{12}^{1} + D_{23}^{2} T_{22}^{1} + D_{33}^{2} T_{32}^{1})$$

$$- C_{20} (D_{13}^{3} T_{12}^{1} + D_{23}^{3} T_{22}^{1} + D_{33}^{3} T_{32}^{1})$$

$$- C_{21} (D_{11}^{3} T_{12}^{1} + D_{21}^{3} T_{22}^{1} + D_{31}^{3} T_{32}^{1})$$

$$+ C_{25} \Big[(D_{13}^{4} + D_{13}^{6}) T_{12}^{1} + (D_{23}^{4} + D_{23}^{6}) T_{22}^{1}$$

$$+ (D_{33}^{4} + D_{33}^{6}) T_{32}^{1}\Big]$$

$$+ C_{28} \Big[(D_{11}^{5} + D_{11}^{7}) T_{12}^{1} + (D_{21}^{5} + D_{21}^{7}) T_{22}^{1}$$

$$+ (D_{31}^{5} + D_{31}^{7}) T_{32}^{1}\Big]$$

$$+ C_{32} \Big[(D_{11}^{8} + D_{11}^{10}) T_{12}^{1} + D_{21}^{8} + D_{21}^{10}) T_{22}^{1}$$

$$+ (D_{31}^{8} + D_{31}^{10}) T_{32}^{1}\Big]$$

$$- C_{34} \Big[(D_{13}^{9} + D_{13}^{11}) T_{12}^{1} + D_{23}^{9} + D_{23}^{11}) T_{22}^{1}$$

$$+ (D_{33}^{9} + D_{33}^{11}) T_{32}^{1}\Big]$$

$$- C_{33} \Big[(D_{11}^{8} - D_{11}^{10}) T_{13}^{1} + (D_{21}^{8} - D_{21}^{10}) T_{23}^{1}$$

$$+ (D_{31}^{8} - D_{31}^{10}) T_{33}^{1}\Big]$$

$$+ C_{35} \Big[(D_{13}^{9} - D_{13}^{1}) T_{13}^{1} + (D_{23}^{9} - D_{23}^{11}) T_{23}^{1}$$

$$+ (D_{33}^{9} - D_{33}^{11}) T_{33}^{1}\Big]$$

$$B\,(7) = (C_3 - C_5)\,\dot{\theta}_2^{\,2} \sin\theta_2 \sin\theta_2 \cos\phi_2$$

$$- 2\dot{\psi}_2\dot{\theta}_2 \sin\theta_2 \cos\theta_2\,(C_3 \cos^2\phi_2 + C_5 \sin^2\phi_2$$

$$+ C_{39} - C_{40})$$

$$- 2\dot{\psi}_2\dot{\phi}_2\, T_{32}{}^2 T_{33}{}^2\,(C_3 - C_5)$$

$$+ \dot{\theta}_2\dot{\phi}_2 \cos\theta_2\,(2\,C_3 \sin^2\phi_2 + 2\,C_5 \cos^2\phi_2 + C_{39})$$

$$- C_{19}\,(D_{13}{}^1 T_{23}{}^2 - D_{23}{}^1 T_{13}{}^2)$$

$$- C_{22}\,(D_{13}{}^3 T_{23}{}^2 - D_{21}{}^3 T_{13}{}^2)$$

$$- C_{23}\,(D_{11}{}^3 T_{23}{}^2 - D_{21}{}^3 T_{13}{}^2)$$

$$+ C_{26}\left[(D_{13}{}^4 + D_{13}{}^6)\,T_{23}{}^2 - (D_{23}{}^4 + D_{23}{}^6)\,T_{13}{}^2\right]$$

$$- C_{27}\left[(D_{13}{}^4 - D_{13}{}^6)\,T_{22}{}^2 - (D_{23}{}^4 - D_{23}{}^6)\,T_{12}{}^2\right]$$

$$- C_{29}\left[(D_{11}{}^5 + D_{11}{}^7)\,T_{23}{}^2 - (D_{21}{}^5 + D_{21}{}^7)\,T_{13}{}^2\right]$$

$$+ C_{30}\left[(D_{11}{}^5 - D_{11}{}^7)\,T_{22}{}^2 - (D_{21}{}^5 - D_{21}{}^7)\,T_{12}{}^2\right]$$

$$
\begin{aligned}
B\,(8) = {} & (C_3 \cos^2 \phi_2 + C_5 \sin^2 \phi_2 + C_{39} - C_{40})\, \dot{\psi}_2^{\,2} \sin \theta_2 \cos \theta_2 \\
& - (2\, C_3 \cos^2 \phi_2 + 2\, C_5 \sin^2 \phi_2 + C_{39})\, \dot{\psi}_2 \dot{\phi}_2 \cos \theta_2 \\
& + 2\, (C_3 - C_5)\, \dot{\theta}_2 \dot{\phi}_2 \sin \phi_2 \cos \phi_2 \\
& + \{ C_{19}\, ({D_{13}}^1 {T_{11}}^2 + {D_{23}}^1 {T_{21}}^2 + {D_{33}}^1 {T_{31}}^2) \\
& + C_{22}\, ({D_{13}}^3 {T_{11}}^2 + {D_{23}}^3 {T_{21}}^2 + {D_{33}}^3 {T_{31}}^2) \\
& + C_{23}\, ({D_{11}}^3 {T_{11}}^2 + {D_{21}}^3 {T_{21}}^3 + {D_{31}}^3 {T_{31}}^2) \\
& - C_{26} \Big[({D_{13}}^4 + {D_{13}}^6)\, {T_{11}}^2 + ({D_{23}}^4 + {D_{23}}^6)\, {T_{21}}^2 \\
& + ({D_{33}}^4 + {D_{33}}^6)\, {T_{31}}^2 \Big] \\
& + C_{29} \Big[({D_{11}}^5 + {D_{11}}^7)\, {T_{11}}^2 + ({D_{21}}^5 + {D_{21}}^7)\, {T_{21}}^2 \\
& + ({D_{31}}^5 + {D_{31}}^7)\, {T_{31}}^2 \Big] \} \cos \phi_2 \\
& + \{ C_{27} \Big[({D_{13}}^4 - {D_{13}}^6)\, {T_{11}}^2 + ({D_{23}}^4 - {D_{23}}^6)\, {T_{21}}^2 \\
& + ({D_{33}}^4 - {D_{33}}^6)\, {T_{31}}^2 \Big] \\
& - C_{30} \Big[({D_{11}}^5 - {D_{11}}^7)\, {T_{11}}^2 + ({D_{21}}^5 - {D_{21}}^7)\, {T_{21}}^2 \\
& + ({D_{31}}^5 - {D_{31}}^7)\, {T_{31}}^2 \Big] \} \sin \phi_2
\end{aligned}
$$

$$
\begin{aligned}
B\ (9) = {} & (C_3 - C_5)\ (\dot{\psi}_2{}^2 \cos^2 \theta_2 + \dot{\theta}_2{}^2) \sin \phi_2 \cos \phi_2 \\
& + (2\, C_3 \cos^2 \phi_2 + 2\, C_5 \sin^2 \phi_2 + C_{39}) \dot{\psi}_2 \dot{\theta}_2 \cos \theta_2 \\
& - C_{19}\ (D_{13}{}^1 T_{12}{}^2 + D_{23}{}^3 T_{22}{}^2 + D_{33}{}^3 T_{32}{}^2) \\
& - C_{22}\ (D_{13}{}^3 T_{12}{}^2 + D_{21}{}^3 T_{22}{}^2 + D_{31}{}^3 T_{32}{}^2) \\
& + C_{26}\ \Big[(D_{13}{}^4 + D_{13}{}^6)\ T_{12}{}^2 + (D_{23}{}^4 + D_{23}{}^6)\ T_{22}{}^2 \\
& + (D_{33}{}^4 + D_{33}{}^6)\ T_{32}{}^2\Big] \\
& + C_{27}\ \Big[(D_{13}{}^4 - D_{13}{}^6)\ T_{13}{}^2 + (D_{23}{}^4 + D_{23}{}^6)\ T_{23}{}^2 \\
& + (D_{33}{}^4 - D_{33}{}^6)\ T_{33}{}^2\Big] \\
& - C_{29}\ \Big[(D_{11}{}^5 + D_{11}{}^7)\ T_{12}{}^2 + (D_{21}{}^5 + D_{21}{}^7)\ T_{22}{}^2 \\
& + (D_{31}{}^5 + D_{31}{}^7)\ T_{32}{}^2\Big] \\
& - C_{30}\ \Big[(D_{11}{}^5 - D_{11}{}^7)\ T_{13}{}^2 + (D_{21}{}^5 - D_{21}{}^7)\ T_{23}{}^2 \\
& + (D_{31}{}^5 - D_{31}{}^7)\ T_{33}{}^2\Big]
\end{aligned}
$$

$$B\ (10) = -\left(C_{13}\rho_3 + C_{41} - C_{42}\right)\Big[\dot{\theta}_3{}^2 T_{31}{}^3 \sin\phi_3 \cos\phi_3$$

$$+ 2\ \dot{\psi}_3 T_{33}{}^3\ (\dot{\theta}_3 \sin\theta_3 \cos\phi_3 + \dot{\phi}_3 \cos\theta_3 \sin\phi_3)\Big]$$

$$+ \Big[(2\ C_{13}\rho_3 + 2\ C_{41} - C_{42}) \sin^2\phi_3$$

$$+ C_{42} \cos^2\phi_3\Big] \cos\theta_3\dot{\theta}_3\dot{\phi}_3 + 2\ C_4\dot{\psi}_3\dot{\theta}_3 \sin\theta_3 \cos\theta_3$$

$$- C_{20}\ (D_{13}{}^1 T_{23}{}^3 - D_{23}{}^1 T_{13}{}^3)$$

$$- C_{21}\ (D_{13}{}^1 T_{21}{}^3 - D_{23}{}^1 T_{11}{}^3)$$

$$- C_{22}\ (D_{13}{}^2 T_{23}{}^3 - D_{23}{}^2 T_{13}{}^3)$$

$$- C_{23}\ (D_{13}{}^2 T_{21}{}^3 - D_{23}{}^2 T_{11}{}^3)$$

$$- C_{24}\ (D_{13}{}^3 T_{21}{}^3 - D_{23}{}^3 T_{11}{}^3)$$

$$- C_{24}\ (D_{11}{}^3 T_{23}{}^3 - D_{21}{}^3 T_{13}{}^3)$$

$$B\ (11) = \Big[(C_{13}\rho_3 + C_{41} - C_{42}) \cos^2\phi_3 - C_4\Big]\ \dot{\psi}_3{}^2 \sin\theta_3 \cos\theta_3$$

$$- \Big[(2\ C_{13}\rho_3 + 2\ C_{41} - C_{42}) \cos^2\phi_3$$

$$+ C_{42} \sin^2\phi_3\Big]\ \dot{\psi}_3\dot{\phi}_3 \cos\theta_3$$

$$+ 2\ (C_{13}\rho_3 + C_{41} - C_{42})\ \dot{\theta}_3\dot{\phi}_3 \sin\phi_3 \cos\phi_3$$

$$+\Big[C_{20}\ (D_{13}{}^{1}T_{11}{}^{3} + D_{23}{}^{1}T_{21}{}^{3} + D_{33}{}^{1}T_{31}{}^{3})$$

$$+ C_{22}\ (D_{13}{}^{2}T_{11}{}^{3} + D_{23}{}^{2}T_{21}{}^{3} + D_{33}{}^{1}T_{31}{}^{3})$$

$$+ C_{24}\ (D_{11}{}^{3}T_{11}{}^{3} + D_{21}{}^{3}T_{21}{}^{3} + D_{31}{}^{3}T_{31}{}^{3})\Big] \cos\phi_3$$

$$- C_{21}\Big[(D_{13}{}^{1}\cos\psi_3 + D_{23}{}^{1}\sin\psi_3)\sin\theta_3 + D_{33}{}^{1}\cos\theta_3\Big]$$

$$- C_{23}\Big[(D_{13}{}^{2}\cos\psi_3 + D_{23}{}^{2}\sin\psi_3)\sin\theta_3 + D_{33}{}^{2}\cos\theta_3\Big]$$

$$- C_{24}\Big[(D_{13}{}^{3}\cos\psi_3 + D_{23}{}^{3}\sin\psi_3)\sin\theta_3 + D_{33}{}^{3}\cos\theta_3\Big]$$

$$B\ (12) = (C_{13}\rho_3 + C_{41} - C_{42})\ (\dot{\psi}_3{}^2\cos^2\theta_3 - \dot{\theta}_3{}^2)\sin\phi_3\cos\phi_3$$

$$+ \Big[(2\ C_{13}\rho_3 + 2\ C_{41} - C_{42})\cos^2\phi_3$$

$$+ C_{42}\sin^2\phi_3\Big]\dot{\psi}_3\dot{\theta}_3\cos\theta_3$$

$$- C_{20}\ (D_{13}{}^{1}T_{12}{}^{3} + D_{23}{}^{1}T_{22}{}^{3} + D_{33}{}^{1}T_{32}{}^{3})$$

$$- C_{22}\ (D_{13}{}^{2}T_{12}{}^{3} + D_{23}{}^{2}T_{22}{}^{3} + D_{33}{}^{2}T_{32}{}^{3})$$

$$- C_{24}\ (D_{11}{}^{3}T_{12}{}^{3} + D_{21}{}^{3}\ T_{22}{}^{3} + D_{31}{}^{3}T_{32}{}^{3})$$

$$
\begin{aligned}
B\ (13) = &\Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45} \\
&+ C_{46})\ \cos^2\ \alpha_5\Big]\ \dot{\theta}_4{}^2\ \sin\ \theta_4\ \sin\ \phi_4\ \cos\ \phi_4 \\
&+ C_8\dot{\theta}_4{}^2\ (\sin^2\ \theta_4 - \cos^2\theta_4)\ \sin\ \psi_4\ \cos\ \psi_4\ \sin^2\phi_4\ \cos^2\alpha_5 \\
&- (C_8 - C_{45} + C_{46})\ (\dot{\theta}_4{}^2 - \dot{\phi}_4{}^2)\ \cos\ \theta_4\ \sin\ \phi_4\ \sin\ \alpha_5\ \cos\ \alpha_5 \\
&- \Big\{\Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45})\ \cos^2\ \alpha_5 \\
&- C_{46}\ \sin^2\ \alpha_5\Big]\ \cos^2\ \phi_4 - (C_8 - C_{45})\ \sin^2\ \alpha_5 \\
&- C_{46}\ \sin^2\ \phi_4 + C_{46}\ \cos^2\ \alpha_5\Big\}\ 2\ \dot{\psi}_4\dot{\theta}_4\ \sin\ \theta_4\ \cos\ \theta_4 \\
&+ 2\ (C_8 - C_{45} + C_{46})\ \dot{\psi}_4\dot{\theta}_4\ (\cos^2\ \theta_4 \\
&- \sin^2\ \theta_4)\ \cos\ \phi_4\ \sin\ \alpha_5\ \cos\ \alpha_5 \\
&- 2\Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45} \\
&+ C_{46})\ \cos^2\ \alpha_5\Big]\ \dot{\psi}_4\dot{\phi}_4\ \cos^2\ \theta_4\ \sin\ \phi_4\ \cos\ \phi_4 \\
&- 2(C_8 - C_{45} + C_{46})\dot{\psi}_4\dot{\phi}_4\ \sin\ \theta_4\ \cos\ \theta_4\ \sin\ \phi_4\ \sin\ \alpha_4\ \cos\ \alpha_5 \\
&+ 2\Big\{\Big[C_8\ \sin^2\ \theta_4 + (C_{45} - C_{46})\ \cos^2\ \theta_4\Big]\ \cos^2\phi_4 - C_8\ \cos^2\theta_4 \\
&+ C_8\ \sin^2\phi_4 - (C_{45} - C_{46})\ \sin^2\theta_4\Big\}\ \dot{\psi}_4\dot{\alpha}_5\ \sin\ \alpha_5\ \cos\ \alpha_5 \\
&+ 2\ (C_8 - C_{45} + C_{46})\ \dot{\psi}_4\dot{\alpha}_5\ (\cos^2\alpha_5 \\
&- \sin^2\alpha_5)\ \sin\ \theta_4\ \cos\ \theta_4\ \cos\ \phi_4
\end{aligned}
$$

$$+ 2C_8\dot{\theta}_4\dot{\phi}_4 \sin\psi_4 \cos\psi_4 \sin\theta_4 \cos\theta_4 \sin\phi_4 \cos\phi_4 \cos^2\alpha_5$$

$$+ \dot{\theta}_4\dot{\phi}_4 \cos\theta_4 \{(2\,C_7 + 2\,C_{43} - C_{44}) \sin^2\phi_4 + C_{44} \cos^2\phi_4$$

$$+ \left[(2\,C_8 - C_{45} + 2\,C_{46}) \sin^2\phi_4 + C_{45} \cos^2\phi_4\right] \cos^2\alpha_5$$

$$+ C_{45} \sin^2\alpha_5\}$$

$$- 2\,C_8\dot{\theta}_4\dot{\alpha}_5 \sin\psi_4 \cos\psi_4 \sin\theta_4 \cos\theta_4 \sin^2\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \dot{\theta}_4\dot{\alpha}_5 \sin\theta_4 \sin\phi_4 \left[(2\,C_8 - C_{45}) \cos^2\alpha_5 + (C_{45}\right.$$

$$\left.- 2\,C_{46}) \sin^2\alpha_5\right]$$

$$+ 2\,(C_8 - C_{45} + C_{46})\, \dot{\theta}_4\dot{\alpha}_5 \cos\theta_4 \sin\phi_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ \dot{\phi}_4\dot{\alpha}_5 \cos\theta_4 \cos\phi_4 \left[(2\,C_8 - C_{45}) \sin^2\alpha_5 + (C_{45}\right.$$

$$\left.- 2\,C_{46}) \cos^2\alpha_5\right]$$

$$- 2\,(C_8 - C_{45} + C_{46})\, \dot{\phi}_4\dot{\alpha}_5 \sin\theta_4 \sin\alpha_5 \cos\alpha_5$$

$$+ C_{25}\,(D_{13}{}^1T_{23}{}^4 - D_{23}{}^1T_{13}{}^4)$$

$$+ C_{26}\,(D_{13}{}^2T_{23}{}^4 - D_{23}{}^2T_{13}{}^4)$$

$$- C_{27}\,(D_{12}{}^2T_{23}{}^4 - D_{22}{}^2T_{13}{}^4)$$

$$- C_{28}\,(D_{13}{}^1T_{21}{}^5 - D_{23}{}^1T_{11}{}^5)$$

$$- C_{29} \ (D_{13}{}^{2}T_{21}{}^{5} - D_{23}{}^{2}T_{11}{}^{5})$$

$$+ C_{30} \ (D_{12}{}^{2}T_{11}{}^{5} - D_{22}{}^{2}T_{11}{}^{5})$$

$$+ C_{31} \ (D_{13}{}^{4}T_{11}{}^{5} - D_{23}{}^{4}T_{11}{}^{5})$$

$$+ C_{31} \ (D_{11}{}^{5}T_{23}{}^{4} - D_{21}{}^{5}T_{13}{}^{4})$$

$$B\ (14) = C_7 \ (\dot{\psi}_4{}^2 \sin\theta_4 \cos\theta_4 \cos^2\phi_4$$

$$+ 2\ \dot{\theta}_4\dot{\phi}_4 \sin\phi_4 \cos\phi_4$$

$$- 2\ \dot{\psi}_4\dot{\phi}_4 \cos\theta_4 \cos^2\phi_4)$$

$$+ C_8 \ (-\dot{\psi}_4{}^2 \sin\theta_4 \cos\theta_4 \sin^2\alpha_5$$

$$- \dot{\psi}_4{}^2 \cos^2\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ \dot{\psi}_4{}^2 \sin^2\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ \dot{\psi}_4{}^2 \sin\theta_4 \cos\theta_4 \cos^2\phi_4 \cos^2\alpha_5$$

$$+ \dot{\psi}_4{}^2 \cos^2\psi_4 \sin\theta_4 \cos\theta_4 \sin^2\phi_4 \cos^2\alpha_5$$

$$- \dot{\psi}_4{}^2 \sin^2\psi_4 \sin\theta_4 \cos\theta_4 \sin^2\phi_4 \cos^2\alpha_4$$

$$+ \dot{\phi}_4{}^2 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- 2\ \dot{\psi}_4\dot{\phi}_4\ \sin\ \theta_4\ \cos\ \phi_4\ \sin\ \alpha_5\ \cos\ \alpha_5$$

$$- 2\ \dot{\psi}_4\dot{\phi}_4\ \cos\ \theta_4\ \cos^2\phi_4\ \cos^2\alpha_5$$

$$+ 2\ \dot{\psi}_4\dot{\alpha}_5\ \sin\ \theta_4\ \sin\ \phi_4\ \sin^2\alpha_5$$

$$+ 2\ \dot{\psi}_4\dot{\phi}_4\ \sin\ \psi_4\ \cos\ \psi_4\ \sin\ \theta_4\ \cos\ \theta_4\ \sin\ \phi_4\ \cos\ \phi_4\ \cos^2\alpha_5$$

$$- 2\ \dot{\psi}_4\dot{\alpha}_5\ \sin\ \psi_4\ \cos\ \psi_4\ \sin\ \theta_4\ \cos\ \theta_4\ \sin^2\phi_4\ \sin\ \alpha_5\ \cos\ \alpha_5$$

$$+ 2\ \dot{\psi}_4\dot{\alpha}_5\ \cos\ \theta_4\ \sin\ \phi_4\ \cos\ \phi_4\ \sin\ \alpha_5\ \cos\ \alpha_5$$

$$+ 2\ \dot{\theta}_4\dot{\phi}_4\ \sin\ \phi_4\ \cos\ \phi_4\ \cos^2\alpha_5$$

$$- 2\ \dot{\theta}_4\dot{\alpha}_5\ \sin^2\phi_4\ \sin\ \alpha_5\ \cos\ \alpha_5$$

$$- 2\ \dot{\phi}_4\dot{\alpha}_5\ \sin\ \phi_4\ \sin^2\alpha_5)$$

$$+ C_{43}\ (\ \dot{\psi}_4^{\ 2}\ \sin\ \theta_4\ \cos\ \theta_4\ \cos^2\phi_4\ -\ 2\ \dot{\psi}_4\dot{\phi}_4\ \cos\ \theta_4\ \cos^2\phi_4$$

$$+ 2\ \dot{\theta}_4\dot{\phi}_4\ \sin\ \phi_4\ \cos\ \phi_4)\ +\ C_{44}\ (\ -\ \dot{\psi}_4^{\ 2}\ \sin\ \theta_4\ \cos\ \theta_4\ \cos^2\phi_4$$

$$+ \dot{\psi}_4\dot{\phi}_4\ \cos\ \theta_4\ \cos^2\phi_4\ -\ \dot{\psi}_4\dot{\phi}_4\ \cos\ \theta_4\ \sin^2\phi_4$$

$$- 2\ \dot{\theta}_4\dot{\phi}_4\ \sin\ \phi_4\ \cos\ \phi_4)$$

$$+ C_{45}\ (\dot{\psi}_4^{\ 2}\ \sin\ \theta_4\ \cos\ \theta_4\ \sin^2\alpha_5$$

$$- \dot{\psi}_4^{\ 2}\ \sin\ \theta_4\ \cos\ \theta_4\ \cos^2\phi_4\ \cos^2\alpha_5$$

$$+ \dot{\psi}_4^2 \cos^2\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \dot{\psi}_4^2 \sin^2\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \dot{\phi}_4^2 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ 2\, \dot{\psi}_4\dot{\phi}_4 \sin\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \dot{\psi}_4\dot{\phi}_4 \cos\theta_4 \sin^2\alpha_5$$

$$- \dot{\psi}_4\dot{\phi}_4 \cos\theta_4 \sin^2\phi_4 \cos^2\alpha_5$$

$$+ \dot{\psi}_4\dot{\phi}_4 \cos\theta_4 \cos^2\phi_4 \cos^2\alpha_5$$

$$- 2\, \dot{\psi}_4\dot{\alpha}_5 \cos\theta_4 \sin\phi_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ \dot{\psi}_4\dot{\alpha}_5 \sin\theta_4 \sin\phi_4 \cos^2\alpha_5$$

$$- \dot{\psi}_4\dot{\alpha}_5 \sin\theta_4 \sin\phi_4 \sin^2\alpha_5 + 2\dot{\theta}_4\dot{\phi}_4 \sin\phi_4 \cos\phi_4 \cos^2\alpha_5$$

$$+ 2\, \dot{\theta}_4\dot{\alpha}_5 \sin^2\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \dot{\phi}_4\dot{\alpha}_5 \sin\phi_4 \cos^2\alpha_5$$

$$+ \dot{\phi}_4\dot{\alpha}_5 \sin\phi_4 \sin^2\alpha_5)$$

$$+ C_{46}\, (-\dot{\psi}_4^2 \sin\theta_4 \cos\theta_4 \sin^2\phi_4$$

$$+ \dot{\psi}_4^2 \sin\theta_4 \cos\theta_4 \cos^2\alpha_5$$

$$- \dot{\psi}_4^{\,2} \sin \theta_4 \cos \theta_4 \cos^2\phi_4 \sin^2\alpha_5$$

$$- \dot{\psi}_4^{\,2} \cos^2\theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ \dot{\psi}_4^{\,2} \sin^2\theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ \dot{\phi}_4^{\,2} \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$- 2 \dot{\psi}_4\dot{\phi}_4 \sin \theta_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$- 2 \dot{\psi}_4\dot{\phi}_4 \cos \theta_4 \cos^2\phi_4 \cos^2\alpha_5$$

$$+ 2 \dot{\psi}_4\dot{\alpha}_5 \cos \theta_4 \sin \phi_4 \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$- 2 \dot{\psi}_4\dot{\alpha}_5 \sin \theta_4 \sin \phi_4 \cos^2\alpha_5$$

$$+ 2 \dot{\theta}_4\dot{\phi}_4 \sin \phi_4 \cos \phi_4 \cos^2\alpha_5$$

$$- 2 \dot{\theta}_4\dot{\alpha}_5 \sin^2\phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ 2 \dot{\phi}_4\dot{\alpha}_5 \sin \phi_4 \cos^2\alpha_5)$$

$$- C_{25} ({D_{13}}^1{T_{11}}^4 + {D_{23}}^1{T_{21}}^4 + {D_{33}}^1{T_{31}}^4) \cos \phi_4$$

$$- C_{26} ({D_{13}}^2{T_{11}}^4 + {D_{23}}^2{T_{21}}^4 + {D_{33}}^2{T_{31}}^4) \cos \phi_4$$

$$+ C_{27} ({D_{12}}^2{T_{11}}^4 + {D_{22}}^2{T_{21}}^4 + {D_{32}}^2{T_{31}}^4) \cos \phi_4$$

$$+ C_{28} \Big[({D_{13}}^1 \cos \psi_4 + {D_{23}}^1 \sin \psi_4) {T_{31}}^5$$

$$+ D_{33}^{1} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5)\Big]$$

$$+ C_{29} \Big[(D_{13}^{2} \cos \psi_4 + D_{23}^{2} \sin \psi_4) T_{31}^{5}$$

$$+ D_{33}^{2} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5)\Big]$$

$$- C_{30} \Big[(D_{12}^{2} \cos \psi_4 + D_{22}^{2} \sin \psi_4) T_{31}^{5}$$

$$+ D_{32}^{2} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5)\Big]$$

$$- C_{31} \Big[(D_{13}^{4} \cos \psi_4 + D_{23}^{2} \sin \psi_4) T_{31}^{5}$$

$$+ D_{33}^{4} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5)\Big]$$

$$- C_{31} (D_{11}^{5} T_{11}^{4} + D_{21}^{5} T_{21}^{4} + D_{31}^{5} T_{31}^{4}) \cos \phi_4$$

$$B\ (14) = \Big\{\Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45}) \cos^2 \alpha_5$$

$$- C_{46} \sin^2 \alpha_5\Big] \cos^2\phi_4$$

$$- (C_8 - C_{45}) \sin^2\alpha_5 + C_{46} (\cos^2\alpha_5$$

$$- \sin^2\phi_4)\Big\} \dot{\psi}_4^{2} \sin \theta_4 \cos \theta_4$$

$$+ (C_8 - C_{45} + C_{46}) (\sin^2\theta_4$$

$$- \cos^2\theta_4) \dot{\psi}_4^{2} \cos \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ C_8 (\cos^2\psi_4 - \sin^2\psi_4)\ \dot{\psi}_4^2 \sin\theta_4 \cos\theta_4 \sin^2\phi_4 \cos^2\alpha_5$$

$$+ (C_8 - C_{45} + C_{46})\ \dot{\phi}_4^2 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \Big\{\Big[2 C_7 + C_{43} - C_4 + (2 C_8 - C_{45}$$

$$+ 2 C_{46}) \cos^2\alpha_5\Big] \cos^2\phi_4 + C_{45} \sin^2\alpha_5$$

$$+ (C_{45} \cos^2\alpha_5 + C_{44}) \sin^2\phi_4\Big\}\ \dot{\psi}_4\dot{\phi}_4 \cos\theta_4$$

$$- 2 (C_8 - C_{45} + C_{46})\ \dot{\psi}_4\dot{\phi}_4 \sin\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ 2C_8\dot{\psi}_4\dot{\phi}_4 \sin\psi_4 \cos\psi_4 \sin\theta_4 \cos\theta_4 \sin\phi_4 \cos\phi_4 \cos^2\alpha_5$$

$$- 2C_8\dot{\psi}_4\dot{\alpha}_5 \sin\psi_4 \cos\psi_4 \sin\theta_4 \cos\theta_4 \sin^2\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ 2 (C_8 - C_{45} + C_{46})\ \dot{\psi}_4\dot{\alpha}_5 \cos\theta_4 \sin\phi_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$+ \Big[(2C_8 - C_{45}) \sin^2\alpha_5 + (C_{45} - 2C_{46}) \cos^2\alpha_5\Big]\ \dot{\psi}_4\dot{\alpha}_5 \sin\theta_4 \sin\phi_4$$

$$+ 2 \Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45} + C_{46}) \cos^2\alpha_5\Big]\ \dot{\theta}_4\dot{\phi}_4 \sin\phi_4 \cos\phi_4$$

$$- 2 (C_8 - C_{45} + C_{46})\ \dot{\theta}_4\dot{\alpha}_5 \sin^2\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- \Big[(2C_8 - C_{45}) \sin^2\alpha_5 + (C_{45} - 2C_{46}) \cos^2\alpha_5\Big]\ \dot{\phi}_4\dot{\alpha}_5 \sin\phi_4$$

$$- \cos\phi_4 \Big[C_{25} (D_{13}{}^1T_{11}{}^4 + D_{23}{}^1T_{21}{}^4 + D_{33}{}^1T_{31}{}^4)$$

$$+ C_{26} (D_{13}{}^2T_{11}{}^4 + D_{23}{}^2T_{21}{}^4 + D_{33}{}^2T_{31}{}^4)$$

$$- C_{27} \left(D_{12}{}^{2}T_{11}{}^{4} + D_{22}{}^{2}T_{21}{}^{4} + D_{32}{}^{2}T_{31}{}^{4}\right)\Big]$$

$$+ T_{31}{}^{5} \Big[C_{28} \left(D_{13}{}^{1} \cos \psi_4 + D_{23}{}^{1} \sin \psi_4\right)$$

$$+ C_{29} \left(D_{13}{}^{2} \cos \psi_4 + D_{23}{}^{1} \sin \psi_4\right)$$

$$- C_{30} \left(D_{12}{}^{2} \cos \psi_4 + D_{23}{}^{2} \sin \psi_4\right)\Big]$$

$$+ \left(C_{28}D_{33}{}^{1} + C_{29}D_{33}{}^{2} - C_{30}D_{32}{}^{2}\right) (\sin \theta_4 \cos \phi_4 \cos \alpha_5$$

$$- \cos \theta_4 \sin \alpha_5)$$

$$- C_{31} \Big[\left(D_{13}{}^{4} \cos \psi_4 + D_{23}{}^{4} \sin \psi_4\right) T_{31}{}^{5}$$

$$+ D_{33}{}^{4} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5)$$

$$+ \left(D_{11}{}^{5}T_{11}{}^{4} + D_{21}{}^{5}T_{21}{}^{4} + D_{31}{}^{5}T_{31}{}^{4}\right) \cos \phi_4\Big]$$

$$B\ (15) = \Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45}$$

$$+ C_{46}) \cos^2\alpha_5\Big] \left(\dot{\psi}_4{}^{2} \cos^2\theta_4 - \dot{\theta}_4{}^{2}\right) \sin \phi_4 \cos \phi_4$$

$$+ (C_8 - C_{45} + C_{46})\ \dot{\psi}_4{}^{2} \sin \theta_4 \cos \theta_4 \sin \phi_4 \sin \alpha_5 \cos \alpha_5$$

$$+ \Big\{\Big[2\ C_7 + 2\ C_{43} - C_{44} + (2\ C_8 - C_{45}$$

$$+ 2\ C_{46}) \cos^2\alpha_5\Big] \cos^2\phi_4 + (C_{44} + C_{45} \cos^2\alpha_5) \sin^2\phi_4$$

$$+ C_{45} \sin^2\alpha_5 \Big\} \dot{\psi}_4 \dot{\theta}_4 \cos\theta_4$$

$$+ 2\,(C_8 - C_{45} + C_{46})\,\dot{\psi}_4\dot{\theta}_4 \sin\theta_4 \cos\phi_4 \sin\alpha_5 \cos\alpha_5$$

$$- 2\,C_8\dot{\psi}_4\dot{\theta}_4 \sin\psi_4 \cos\psi_4 \sin\theta_4 \cos\theta_4 \sin\phi_4 \cos\phi_4 \cos^2\alpha_5$$

$$+ 2\,(C_8 - C_{45} + C_{46})\,(\dot{\phi}_4 - \dot{\psi}_4 \sin\theta_4)\dot{\alpha}_5 \sin\alpha_5 \cos\alpha_5$$

$$+ \Big[(2\,C_8 - C_{45}) \cos^2\alpha_5 + (C_{45} - 2\,C_{46}) \sin^2\alpha_5\Big]$$

$$\times\,(\dot{\theta}_4\dot{\alpha}_5 \sin\phi_4 - \dot{\psi}_4\dot{\alpha}_5 \cos\theta_4 \cos\phi_4)$$

$$+ (C_{25} + C_{28}\cos\alpha_5)\,(D_{13}{}^1T_{12}{}^4 + D_{23}{}^1T_{22}{}^4 + D_{33}{}^1T_{32}{}^4)$$

$$+ (C_{26} + C_{29}\cos\alpha_5)\,(D_{13}{}^2T_{12}{}^4 + D_{23}{}^2T_{22}{}^4 + D_{33}{}^2T_{32}{}^4)$$

$$- (C_{27} + C_{30}\cos\alpha_5)\,(D_{12}{}^2T_{12}{}^4 + D_{22}{}^2T_{22}{}^4 + D_{32}{}^2T_{32}{}^4)$$

$$- C_{31}\cos\alpha_5\,(D_{13}{}^4T_{12}{}^4 + D_{23}{}^4T_{22}{}^4 + D_{33}{}^4T_{32}{}^4)$$

$$+ C_{31}\,(D_{11}{}^5T_{12}{}^4 + D_{21}{}^5T_{12}{}^4 + D_{21}{}^5T_{22}{}^4 + D_{31}{}^5T_{32}{}^4)$$

$$B\,(16) = \Big\{\Big[C_8 + (C_{46} - C_{45}) \cos^2\phi_4\Big] \cos^2\theta_4$$

$$- (C_8 \cos^2\phi_4 - C_{45} + C_{46}) \sin^2\theta_4$$

$$- C_8 \sin^2\phi_4 \Big\}\,\dot{\psi}_4{}^2 \sin\alpha_5 \cos\alpha_5$$

$$+ (C_8 - C_{45} + C_{46}) (\sin^2\alpha_5$$

$$- \cos^2\alpha_5) \dot{\psi}_4{}^2 \sin \theta_4 \cos \theta_4 \cos \phi_4$$

$$+ (C_8 - C_{45} + C_{46}) (\dot{\theta}_4{}^2 \sin^2\phi_4 - \dot{\phi}_4{}^2) \sin \alpha_5 \cos \alpha_5$$

$$+ 2 C_8\dot{\psi}_4\dot{\theta}_4 \sin \psi_4 \cos \psi_4 \sin \theta_4 \cos \theta_4 \sin \phi_4 \sin \alpha_5 \cos\alpha_5$$

$$- \Big[(2 C_8 - C_{45}) \sin^2\alpha_5 + (C_{45}$$

$$- 2 C_{46}) \cos^2\alpha_5\Big] \dot{\psi}_4\dot{\theta}_4 \sin \theta_4 \sin \phi_4$$

$$+ \Big[(2 C_8 - C_{45}) \cos^2\alpha_5 + (C_{45}$$

$$- 2 C_{46}) \sin^2\alpha_5\Big] (\dot{\psi}_4 \cos \theta_4 \cos \phi_4 - \dot{\theta}_4 \sin \phi_4) \dot{\phi}_4$$

$$+ C_{28} (D_{13}{}^1T_{13}{}^5 + D_{23}{}^1T_{23}{}^5 + D_{33}{}^1T_{33}{}^5)$$

$$+ C_{29} (D_{13}{}^2T_{13}{}^5 + D_{23}{}^2T_{23}{}^5 + D_{33}{}^2T_{33}{}^5)$$

$$- C_{30} (D_{12}{}^2T_{13}{}^5 + D_{22}{}^2T_{23}{}^5 + D_{32}{}^2T_{33}{}^5)$$

$$- C_{31} (D_{13}{}^4T_{13}{}^5 + D_{23}{}^4T_{23}{}^5 + D_{33}{}^4T_{33}{}^5)$$

$$B\ (17) = \Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45}$$

$$+ C_{46}) \cos^2\alpha_7\Big] \dot{\theta}_6{}^2 \sin \theta_6 \sin \phi_6 \cos \phi_6$$

$$- C_8\dot{\theta}_6^{\,2} (\sin^2\theta_6 - \cos^2\theta_6) \sin \psi_6 \cos \psi_6 \sin^2\phi_6 \cos^2\alpha_7$$

$$- (C_8 - C_{45} + C_{46}) (\dot{\theta}_6^{\,2} - \dot{\phi}_6^{\,2}) T_{32}{}^6 \sin \alpha_7 \cos \alpha_7$$

$$- 2 \left\{\left[C_7 + C_{43} - C_{44} + (C_8 - C_{45}) \cos^2\alpha_7\right.\right.$$

$$\left. - C_{46} \sin^2\alpha_7\right] \cos^2\phi_6 - (C_8 - C_{45}) \sin^2\alpha_7$$

$$\left. - C_{46} (\sin^2\phi_6 - \sin \alpha_7 \cos \alpha_7)\right\} \dot{\psi}_6\dot{\theta}_6 \sin \theta_6 \cos \theta_6$$

$$- 2 (C_8 - C_{45} + C_{46}) \dot{\psi}_6\dot{\theta}_6 (\sin^2\theta_6$$

$$- \cos^2\theta_6) \cos \phi_6 \sin \alpha_7 \cos \alpha_7$$

$$- 2 \left\{\left[C_7 + C_{43} - C_{44} + (C_8 - C_{45} + C_{46}) \cos^2\alpha_7\right] T_{33}{}^6\right.$$

$$\left. + (C_8 - C_{45} + C_{46}) \sin \theta_6 \sin \alpha_7 \cos \alpha_7\right\} \dot{\psi}_6\dot{\phi}_6 T_{32}{}^6$$

$$+ 2 \left\{(C_8 \cos^2\phi_6 - C_{45} + C_{46}) \sin^2\theta_6\right.$$

$$- \left[C_8 - (C_{45} - C_{46}) \cos^2\phi_6\right] \cos^2\theta_6$$

$$\left. + C_8 \sin^2\phi_6\right\} \dot{\psi}_6\dot{\alpha}_7 \sin \alpha_7 \cos \alpha_7$$

$$+ 2 \dot{\psi}_6\dot{\alpha}_7 (C_8 - C_{45} + C_{46}) (\cos^2\alpha_7 - \sin^2\alpha_7) \sin \theta_6 T_{33}{}^6$$

$$+ \left\{(2 C_7 + 2 C_{43} - C_{44}) \sin^2\phi_6 + C_{44} \cos^2\phi_6\right.$$

$$+ \left[(2\ C_8 - C_{45} + 2\ C_{46})\ \sin^2\phi_6 + C_{45}\ \cos^2\phi_6\right] \cos^2\alpha_7$$

$$+ C_{45}\ \sin^2\alpha_7$$

$$+ 2\ C_8\ \sin\psi_6\ \cos\psi_6\ \sin\theta_6\ \sin\phi_6\ \cos\phi_6\ \cos^2\alpha_7 \Big\}\ \dot{\theta}_6\dot{\phi}_6\ \cos\theta_6$$

$$+ \Big\{ 2\ (C_8 - C_{45} + C_{46})\ \cos\theta_6\ \cos\phi_6\ \sin\alpha_7\ \cos\alpha_7$$

$$- \left[(2\ C_8 - C_{45})\ \cos^2\alpha_7 + (C_{45} - 2\ C_{46})\ \sin^2\alpha_7\right] \sin\theta_6$$

$$- 2\ C_8\ \sin\psi_6\ \cos\psi_6\ \sin\theta_6\ \cos\theta_6\ \sin\phi_6\ \sin\alpha_7\ \cos\alpha_7 \Big\}$$

$$\times\ \dot{\theta}_6\dot{\alpha}_7\ \sin\phi_6$$

$$+ \Big\{ \left[(2\ C_8 - C_{45})\ \sin^2\alpha_7 + (C_{45} - 2\ C_{46})\ \cos^2\alpha_7\right] T_{33}{}^6$$

$$+ 2\ (C_8 - C_{45} + C_{46})\ T_{31}{}^6\ \sin\alpha_7\ \cos\alpha_7 \Big\}\ \dot{\phi}_6\dot{\alpha}_7$$

$$+ C_{25}\ (D_{13}{}^1T_{23}{}^6 - D_{23}{}^1T_{13}{}^6) + C_{26}\ (D_{13}{}^2T_{23}{}^6 - D_{23}{}^2T_{13}{}^6)$$

$$+ C_{27}\ (D_{12}{}^2T_{23}{}^6 - D_{22}{}^2T_{13}{}^6) - C_{28}\ (D_{13}{}^1T_{21}{}^7 - D_{23}{}^1T_{11}{}^7)$$

$$- C_{29}\ (D_{13}{}^2T_{21}{}^7 - D_{23}{}^2T_{11}{}^7) - C_{30}\ (D_{12}{}^2T_{21}{}^7 - D_{22}{}^2T_{11}{}^7)$$

$$+ C_{31}\ (D_{13}{}^6T_{21}{}^7 - D_{23}{}^6T_{11}{}^7) + C_{31}\ (D_{11}{}^7T_{23}{}^6 - D_{21}{}^7T_{13}{}^6)$$

$$
\begin{aligned}
B\ (18) = \ & \Big\{\Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45})\ \cos^2\alpha_7 \\
& - C_{46}\ \sin^2\alpha_7\Big]\cos^2\phi_6 - (C_8 - C_{45})\ \sin^2\alpha_7 \\
& - C_{46}\ (\sin^2\phi_6 - \cos^2\alpha_7)\Big\}\ \dot{\psi}_6^{\,2}\ \sin\ \theta_6\ \cos\ \theta_6 \\
& + C_8\dot{\psi}_6^{\,2}\ (\cos^2\psi_6 - \sin^2\psi_6\ )\ \sin\ \theta_6\ \cos\ \theta_6\ \sin^2\phi_6\ \cos^2\alpha_7 \\
& + (C_8 - C_{45} + C_{46})\ \Big[\dot{\psi}_6^{\,2}\ (\sin^2\theta_6 - \cos^2\theta_6) \\
& + \dot{\phi}_6^{\,2}\Big]\ \cos\ \phi_6\ \sin\ \alpha_7\ \cos\ \alpha_7 \\
& - \Big\{\Big[2\ C_7 + 2\ C_{43} - C_{44} + (2\ C_8 - C_{45} \\
& + 2\ C_{46})\ \cos^2\alpha_7\Big]\ \cos^2\phi_6 \\
& + (C_{44} + C_{45}\ \cos^2\alpha_7)\ \sin^2\phi_6 + C_{45}\ \sin^2\alpha_7\Big\}\ \dot{\psi}_6\dot{\phi}_6\ \cos\ \theta_6 \\
& - 2\ \Big[(C_8 - C_{45} + C_{46})\ \sin\ \alpha_7 \\
& - C_8\ \sin\ \psi_6\ \cos\ \psi_6\ \cos\ \theta_6\ \sin\ \phi_6\ \cos\ \alpha_7\Big] \\
& \times\ \dot{\psi}_6\dot{\phi}_6\ \sin\ \theta_6\ \cos\ \phi_6\ \cos\ \alpha_7 \\
& + \Big\{\Big[(2\ C_8 - C_{45})\ \sin^2\alpha_7 + (C_{45} - 2\ C_{46})\ \cos^2\alpha_7\Big]\ \sin\ \theta_6 \\
& + 2\ \Big[(C_8 - C_{45} + C_{46})\ \cos\ \phi_6 \\
& - C_8\ \sin\ \psi_6\ \cos\ \psi_6\ \sin\ \theta_6\ \sin\ \phi_6\Big]\ \cos\ \theta_6\ \sin\ \alpha_7\ \cos\ \alpha_7\Big\}
\end{aligned}
$$

$$\times \dot{\psi}_6 \dot{\alpha}_7 \sin \phi_6$$

$$+ 2 \left[C_7 + C_{43} - C_{44} + (C_8 - C_{45} + C_{46}) \cos^2\alpha_7\right] \dot{\theta}_6 \dot{\phi}_6$$

$$\times \sin \phi_6 \cos \phi_6$$

$$- 2 (C_8 - C_{45} + C_{46}) \dot{\theta}_6 \dot{\alpha}_7 \sin^2\phi_6 \sin \alpha_7 \cos \alpha_7$$

$$- \left[(2 C_8 - C_{45}) \sin^2\alpha_7 + (C_{45} - 2 C_{46}) \cos^2\alpha_7\right] \dot{\phi}_6 \dot{\alpha}_7 \sin \phi_6$$

$$- \cos \phi_6 \left[C_{25} ({D_{13}}^1 {T_{11}}^6 + {D_{23}}^1 {T_{21}}^6 + {D_{33}}^1 {T_{31}}^6)\right.$$

$$+ C_{26} ({D_{13}}^2 {T_{11}}^6 + {D_{23}}^2 {T_{21}}^6 + {D_{33}}^2 {T_{31}}^6)$$

$$\left. + C_{27} ({D_{12}}^2 {T_{11}}^6 + {D_{22}}^2 {T_{21}}^6 + {D_{32}}^2 {T_{31}}^6)\right]$$

$$+ {T_{31}}^7 \left[C_{28} ({D_{13}}^1 \cos \psi_6 + {D_{23}}^1 \sin \psi_6)\right.$$

$$+ C_{29} ({D_{13}}^2 \cos \psi_6 + {D_{23}}^2 \sin \psi_6)$$

$$\left. + C_{30} ({D_{12}}^2 \cos \psi_6 + {D_{22}}^2 \sin \psi_6)\right]$$

$$+ (C_{28}{D_{33}}^1 + C_{29}{D_{33}}^2 + C_{30}{D_{32}}^2) (\sin \theta_6 \cos \phi_6 \cos \alpha_7$$

$$- \cos \theta_6 \sin \alpha_7$$

$$- C_{31} \left[({D_{13}}^6 \cos \psi_6 + {D_{23}}^6 \sin \psi_6) {T_{31}}^7\right.$$

$$+ {D_{33}}^6 (\sin \theta_6 \cos \phi_6 \cos \alpha_7 - \cos \theta_6 \sin \alpha_7)$$

$$\left. + ({D_{11}}^7 {T_{11}}^6 + {D_{21}}^7 {T_{21}}^6 + {D_{31}}^7 {T_{31}}^6) \cos \phi_6\right]$$

$$
\begin{aligned}
B\ (19) = & \Big[C_7 + C_{43} - C_{44} + (C_8 - C_{45} \\
& + C_{46}) \cos^2\alpha_7\Big] (\dot{\psi}_6^{\,2} \cos^2\theta_6 - \dot{\theta}_6^{\,2}) \sin\phi_6 \cos\phi_6 \\
& + (C_8 - C_{45} + C_{46})\, \dot{\psi}_6^{\,2} \sin\theta_6 \cos\theta_6 \sin\phi_6 \sin\alpha_7 \cos\alpha_7 \\
& + \Big\{\Big[2\,C_7 + 2\,C_{43} - C_{44} + (2\,C_8 - C_{45} \\
& + 2\,C_{46}) \cos^2\alpha_7\Big] \cos^2\phi_6 + (C_{44} + C_{45} \cos^2\alpha_7) \sin^2\phi_6 \\
& + C_{45} \sin^2\alpha_7\Big\}\, \dot{\psi}_6 \dot{\theta}_6 \cos\theta_6 \\
& + 2\,\Big[(C_8 - C_{45} + C_{46}) \sin\alpha_7 \\
& - C_8 \sin\psi_6 \cos\psi_6 \cos\theta_6 \sin\phi_6 \cos\alpha_7\Big] \\
& \times \dot{\psi}_6 \dot{\theta}_6 \sin\theta_6 \cos\phi_6 \cos\alpha_7 \\
& + 2\,(C_8 - C_{45} + C_{46})\, (\dot{\phi}_6 - \dot{\psi}_6 \sin\theta_6) \dot{\alpha}_7 \sin\alpha_7 \cos\alpha_7 \\
& + \Big[(2\,C_8 - C_{45}) \cos^2\alpha_7 + (C_{45} \\
& - 2\,C_{46}) \sin^2\alpha_7\Big] (\dot{\theta}_6 \dot{\alpha}_7 \sin\phi_6 - \dot{\psi}_6 \dot{\alpha}_7 \cos\theta_6 \cos\phi_6) \\
& + (C_{25} + C_{28} \cos\alpha_7)\, (D_{13}{}^{1}T_{12}{}^{6} + D_{23}{}^{1}T_{22}{}^{6} + D_{33}{}^{1}T_{32}{}^{6}) \\
& + (C_{26} + C_{29} \cos\alpha_7)\, (D_{13}{}^{2}T_{12}{}^{6} + D_{23}{}^{2}T_{22}{}^{6} + D_{33}{}^{2}T_{32}{}^{6}) \\
& + (C_{27} + C_{30} \cos\alpha_7)\, (D_{12}{}^{2}T_{12}{}^{6} + D_{22}{}^{2}T_{22}{}^{6} + D_{32}{}^{2}T_{32}{}^{6})
\end{aligned}
$$

$$- C_{31} \cos \alpha_7 \; (D_{13}{}^6 T_{12}{}^6 + D_{23}{}^6 T_{22}{}^6 + D_{33}{}^6 T_{32}{}^6)$$

$$+ C_{31} \; (D_{11}{}^7 T_{12}{}^6 + D_{21}{}^7 T_{22}{}^6 + D_{31}{}^7 T_{32}{}^6)$$

$$B\ (20) = (C_8 - C_{45} + C_{46}) \; \dot{\psi}_6{}^2 \; (\sin^2\alpha_7 - \cos^2\alpha_7) \sin \theta_6 \; T_{33}{}^6$$

$$+ \Big\{ \Big[C_8 + (C_{46} - C_{45}) \cos^2\phi_6 \Big] \cos^2\theta_6$$

$$- (C_8 \cos^2\phi_6 - C_{45} + C_{46}) \sin^2\theta_6$$

$$- C_8 \sin^2\phi_6 \Big\} \; \dot{\psi}_6{}^2 \sin \alpha_7 \cos \alpha_7$$

$$+ (C_8 - C_{45} + C_{46}) \; (\dot{\theta}_6{}^2 \sin^2\phi_6 - \dot{\phi}_6{}^2) \sin \alpha_7 \cos \alpha_7$$

$$+ 2 \; \Big\{ \Big[C_8 \sin \psi_6 \cos \psi_6 \sin \theta_6 \sin \phi_6$$

$$- (C_8 - C_{45} + C_{46}) \cos \phi_6 \Big] \; \dot{\theta}_6 \; T_{32}{}^6$$

$$+ (C_8 - C_{45} + C_{46}) \; \dot{\phi}_6 \sin \theta_6 \Big\} \; \dot{\psi}_6 \sin \alpha_7 \cos \alpha_7$$

$$- \Big[(2\; C_8 - C_{45} \sin^2\alpha_7 + (C_{45}$$

$$- 2\; C_{46}) \cos^2\alpha_7 \Big] \; \dot{\psi}_6 \dot{\theta}_6 \sin \theta_6 \cos \phi_6$$

$$+ \Big[(2\; C_8 - C_{45}) \cos^2\alpha_7 + (C_{45} - 2\; C_{46}) \sin^2\alpha_7 \Big] \; (\dot{\psi}_6 T_{33}{}^6$$

$$- \dot{\theta}_6 \sin \phi_6) \; \dot{\phi}_6$$

$$+ C_{28} (D_{13}{}^{1}T_{13}{}^{7} + D_{23}{}^{1}T_{23}{}^{7} + D_{33}{}^{1}T_{33}{}^{7})$$

$$+ C_{29} (D_{13}{}^{2}T_{13}{}^{7} + D_{23}{}^{2}T_{23}{}^{7} + D_{33}{}^{2}T_{33}{}^{7})$$

$$+ C_{30} (D_{12}{}^{2}T_{13}{}^{7} + D_{22}{}^{2}\ T_{23}{}^{7} + D_{32}{}^{2}\ T_{33}{}^{7})$$

$$- C_{31} (D_{13}{}^{6}T_{13}{}^{7} + D_{23}{}^{6}T_{23}{}^{7} + D_{33}{}^{6}T_{33}{}^{7})$$

$$B\ (21) = (C_{10} + C_{49} - C_{50})\ \dot{\theta}_8{}^2 \sin\theta_8 \sin\phi_8 \cos\phi_8 \sin^2\alpha_9$$

$$- (C_{10} - C_{49} - C_{50})\ (\dot{\theta}_8{}^2 - \dot{\phi}_8{}^2)\ T_{32}{}^{8} \sin\alpha_9 \cos\alpha_9$$

$$+ 2 \Big\{ C_9 - C_{47} + C_{48} - \Big[(C_{10} - C_{50}) \cos^2\phi_8$$

$$+ C_{49} \Big] \sin^2\alpha_9 + (C_{10} + C_{49} \cos^2\phi_8$$

$$- C_{50}) \cos^2\alpha_9 + C_{49} \sin^2\phi_8 \Big\}\ \dot{\psi}_8\dot{\theta}_8 \sin\theta_8 \cos\theta_8$$

$$+ 2\ (C_{10} + C_{49} - C_{50})\ (\cos^2\theta_8$$

$$- \sin^2\theta_8)\ \dot{\psi}_8\dot{\theta}_8 \cos\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$- 2\ (C_{10} + C_{49} - C_{50})\ \dot{\psi}_8\dot{\phi}_8\ (\sin\theta_8 \cos\alpha_9$$

$$+ \cos\theta_8 \cos\phi_8 \sin\alpha_9) \cos\theta_8 \sin\phi_8 \sin\alpha_9$$

$$+ 2 \Big\{ \Big[C_{10} + (C_{49} - C_{50}) \cos^2\phi_8 \Big] \cos^2\theta_8$$

$$- (C_{10} \cos^2\phi_8 + C_{49} - C_{50}) \sin^2\theta_8$$

$$- C_{10} \sin^2\phi_8 \Big\} \dot{\psi}_8 \dot{\alpha}_9 \sin \alpha_9 \cos \alpha_9$$

$$+ 2 (C_{10} + C_{49} - C_{50}) (\cos^2\alpha_9 - \sin^2\alpha_9) \dot{\psi}_8 \dot{\alpha}_9 \sin \theta_8 T_{33}{}^8$$

$$+ \Big\{ C_{47} + \Big[(2C_{10} + 2C_{49} - C_{50}) \sin^2\phi_8 + C_{50} \cos^2\phi_8 \Big] \sin^2\alpha_9$$

$$+ C_{50} \cos^2\alpha_9 \Big\} \dot{\theta}_8 \dot{\phi}_8 \cos \theta_8$$

$$+ \Big\{ \Big[(2 C_{10} + 2 C_{49} - C_{50}) \sin^2\alpha_9 + C_{50} \cos^2\alpha_9 \Big] \sin \theta_8$$

$$- 2 (C_{10} + C_{49} - C_{50}) T_{33}{}^8 \sin \alpha_9 \cos \alpha_9 \Big\} \dot{\theta}_8 \dot{\alpha}_9 \sin \phi_8$$

$$+ \Big\{ 2 (C_{10} + C_{49} - C_{50}) \sin \theta_8 \cos \alpha_9 \sin \alpha_9$$

$$- \Big[(2 C_{10} + 2 C_{49} - C_{50}) \cos^2\alpha_9 + C_{50} \sin^2\alpha_9 \Big] T_{33}{}^8 \Big\} \dot{\phi}_8 \dot{\alpha}_9$$

$$+ C_{32} (D_{13}{}^1 T_{21}{}^8 - D_{23}{}^1 T_{11}{}^8)$$

$$+ C_{33} (D_{12}{}^1 T_{21}{}^8 - D_{22}{}^1 T_{11}{}^8)$$

$$- C_{34} (D_{13}{}^1 T_{23}{}^9 - D_{23}{}^1 T_{13}{}^9)$$

$$- C_{35} (D_{12}{}^1 T_{23}{}^9 - D_{22}{}^1 T_{13}{}^9)$$

$$+ C_{36} (D_{11}{}^8 T_{23}{}^9 - D_{21}{}^8 T_{13}{}^9 + D_{13}{}^9 T_{21}{}^8 - D_{23}{}^9 T_{11}{}^8)$$

$$B\ (22) = (C_{10} + C_{49} - C_{50}) \Big[\dot{\psi}_8^{\,2}\ (\sin^2\theta_8 - \cos^2\theta_8)$$

$$+ \dot{\phi}_8^{\,2} \Big] \cos\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$- \Big\{ C_9 - C_{47} + C_{48} + (C_{10} + C_{49} \cos^2\phi_8$$

$$+ C_{49}) \sin^2\alpha_9 + C_{49} \sin^2\phi_8 \Big\}\ \dot{\psi}_8^{\,2} \sin\theta_8 \cos\theta_8$$

$$- 2\ (C_{10} + C_{49} - C_{50})\ \dot{\psi}_8 \dot{\phi}_8 \sin\theta_8 \cos\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$- \Big\{ C_{47} + \Big[(2\ C_{10} + 2\ C_{49} - C_{50}) \cos^2\phi_8$$

$$+ C_{50} \sin^2\phi_8 \Big] \sin^2\alpha_9 + C_{50} \cos^2\alpha_9 \Big\}\ \dot{\psi}_8 \dot{\phi}_8 \cos\theta_8$$

$$- \Big[(2\ C_{10} + 2\ C_{49} - C_{50}) \cos^2\alpha_9 + C_{50} \sin^2\alpha_9 \Big] (\dot{\psi}_8 \sin\theta_8$$

$$- \dot{\phi}_8)\ \dot{\alpha}_9 \sin\phi_8$$

$$+ 2\ (C_{10} + C_{49} - C_{50})\ (\dot{\theta}_8 \sin\phi_8 - \dot{\psi}_8 \cos\theta_8 \cos\phi_8)$$

$$\times\ \dot{\alpha}_9 \sin\phi_8 \sin\alpha_9 \cos\alpha_9$$

$$+ 2\ (C_{10} + C_{49} - C_{50})\ \dot{\theta}_8 \dot{\phi}_8 \sin\phi_8 \cos\phi_8 \sin^2\alpha_9$$

$$+ (C_{32} \sin\theta_8 + C_{34} T_{33}^{\ 9})\ (D_{13}^{\ 1} \cos\psi_8 + D_{23}^{\ 1} \sin\psi_8)$$

$$+ (C_{32} \cos\theta_8 + C_{34} \Big\{ \cos\theta_8 \cos\alpha_9$$

$$- \sin\theta_8 \cos\phi_8 \sin\alpha_9 \Big\}\ )\ D_{33}^{\ 1}$$

$$+ (C_{33} \sin \theta_8 + C_{35} T_{33}{}^9) (D_{12}{}^1 \cos \psi_8 + D_{22}{}^1 \sin \psi_8)$$

$$+ \Big[C_{33} \cos \theta_8 + C_{35} (\cos \theta_8 \cos \alpha_9$$

$$- \sin \theta_8 \cos \phi_8 \sin \alpha_9) \Big] D_{32}{}^1$$

$$+ C_{36} \Big[(D_{13}{}^9 \cos \psi_8 + D_{23}{}^9 \sin \psi_8) \sin \theta_8 + D_{33}{}^9 \cos \theta_8$$

$$- (D_{11}{}^8 \cos \psi_8 + D_{21}{}^8 \sin \psi_8) T_{33}{}^9$$

$$- D_{31}{}^8 (\cos \theta_8 \cos \alpha_9 - \sin \theta_8 \cos \phi_8 \sin \alpha_9) \Big]$$

$$B\ (23) = (C_{10} + C_{49} - C_{50}) \dot{\psi}_8{}^2 \sin \theta_8 T_{32}{}^8 \sin \alpha_9 \cos \alpha_9$$

$$+ (C_{10} + C_{49} - C_{50}) (\dot{\psi}_8{}^2 \cos^2 \theta_8 - \dot{\theta}_8{}^2) \sin \phi_8 \cos \phi_8 \sin^2 \alpha_9$$

$$+ \Big\{ C_{47} + \Big[(2C_{10} + 2C_{49} - C_{50}) \cos^2 \phi_8 + C_{50} \sin^2 \phi_8 \Big] \sin^2 \alpha_9$$

$$+ C_{50} \cos^2 \alpha_9 \Big\} \dot{\psi}_8 \dot{\theta}_8 \cos \theta_8$$

$$+ 2 (C_{10} + C_{49} - C_{50}) (\dot{\theta}_8 \cos \phi_8$$

$$+ \dot{\alpha}_9) \dot{\psi}_8 \sin \theta_8 \sin \alpha_9 \cos \alpha_9$$

$$+ \left[(2\ C_{10} + 2\ C_{49} - C_{50})\ \sin^2\alpha_9 + C_{50}\ \cos^2\alpha_9\right] (\dot{\psi}_8 T_{33}{}^8$$

$$- \dot{\theta}_8 \sin \phi_8)\ \dot{\alpha}_9$$

$$- 2\ (C_{10} + C_{49} - C_{50})\ \dot{\phi}_8 \dot{\alpha}_9 \sin \alpha_9 \cos \alpha_9$$

$$- \left[C_{34}\ (D_{13}{}^1T_{12}{}^8 + D_{23}{}^1T_{22}{}^8 + D_{33}{}^1T_{32}{}^8)\right.$$

$$+ C_{35}\ (D_{12}{}^1T_{12}{}^8 + D_{22}{}^1T_{22}{}^8 + D_{32}{}^1T_{32}{}^8)$$

$$\left.- C_{36}\ (D_{11}{}^8T_{12}{}^8 + D_{21}{}^8T_{22}{}^8 + D_{31}{}^8T_{32}{}^8)\right] \sin \alpha_9$$

$$B\ (24) = (C_{10} + C_{49} - C_{50})\ \dot{\psi}_8{}^2\ (\sin^2\alpha_9 - \cos^2\alpha_9) \sin \theta_8\ T_{33}{}^8$$

$$+ \left\{(C_{10} \cos^2\phi_8 + C_{49} - C_{50}) \sin^2\theta_8 + C_{10} \sin^2\phi_8\right.$$

$$\left.- \left[(C_{49} - C_{50}) \cos^2\phi_8 + C_{10}\right] \cos^2\theta_8\right\} \dot{\psi}_8{}^2 \sin \alpha_9 \cos \alpha_9$$

$$+ (C_{10} + C_{49} - C_{50})\ (\dot{\phi}_8{}^2 - \dot{\theta}_8{}^2 \sin^2\phi_8) \sin \alpha_9 \cos \alpha_9$$

$$+ \left\{\left[(2\ C_{10} + 2\ C_{49} - C_{50}) \cos^2\alpha_9 + C_{50} \sin^2\alpha_9\right] \sin \theta_8\right.$$

$$+ 2\ (C_{10} + C_{49}$$

$$- C_{50}) \cos \theta_8 \cos \phi_8 \sin \alpha_9 \cos \alpha_9\ \dot{\psi}_8 \dot{\theta}_8 \sin \phi_8$$

$$- 2\ (C_{10} + C_{49} - C_{50})\ \dot{\psi}_8 \dot{\phi}_8 \sin \theta_8 \sin \alpha_9 \cos \alpha_9$$

$$+ \left[(2\ C_{10} + 2\ C_{49} - C_{50}) \sin^2\alpha_9 + C_{50} \cos^2\alpha_9\right] (\dot{\theta}_8 \sin \phi_8$$

$$- \dot{\psi}_8 T_{33}{}^8)\ \dot{\phi}_8$$

$$+ C_{34}\ (D_{13}{}^1T_{11}{}^9 + D_{23}{}^1T_{21}{}^9 + D_{33}{}^1T_{31}{}^9)$$

$$+ C_{35}\ (D_{12}{}^1T_{11}{}^9 + D_{22}{}^1T_{21}{}^9 + D_{32}{}^1T_{31}{}^9)$$

$$- C_{36}\ (D_{11}{}^8T_{11}{}^9 + D_{21}{}^8T_{21}{}^9 + D_{31}{}^8T_{31}{}^9)$$

$$B\ (25) = (C_{10} - C_{49} - C_{50})\ \dot{\theta}_{10}{}^2 \sin \theta_{10} \sin \phi_{10} \cos \phi_{10} \sin^2\alpha_{11}$$

$$- (C_{10} + C_{49} - C_{50})\ (\dot{\theta}_{10}{}^2 - \dot{\phi}_{10}{}^2)\ T_{32}{}^{10} \sin \alpha_{11} \cos \alpha_{11}$$

$$+ 2 \Big\{ C_9 - C_{47} + C_{48} - \Big[(C_{10} - C_{50}) \cos^2\phi_{10} + C_{49} \Big] \sin^2\alpha_{11}$$

$$+ (C_{10} + C_{49} \cos^2\phi_{10} - C_{50}) \cos^2\alpha_{11}$$

$$+ C_{49} \sin^2\phi_{10} \Big\}\ \dot{\psi}_{10}\dot{\theta}_{10} \sin \theta_{10} \cos \theta_{10}$$

$$+ 2\ (C_{10} - C_{49} - C_{50})\ \dot{\psi}_{10}\dot{\theta}_{10}\ (\cos^2\theta_{10}$$

$$- \sin^2\theta_{10}) \cos \phi_{10} \sin \alpha_{11} \cos \alpha_{11}$$

$$- 2\ (C_{10} + C_{49} - C_{50})\ \dot{\theta}_{10}\dot{\phi}_{10}\ T_{33}{}^{11}T_{32}{}^{10} \sin \alpha_{11}$$

$$+ 2 \Big\{ \Big[C_{10} + (C_{49} - C_{50}) \cos^2\phi_{10}\Big] \cos^2\theta_{10}$$

$$- (C_{10} \cos^2\phi_{10} + C_{49} - C_{50}) \sin^2\theta_{10}$$

$$- C_{10} \sin^2\phi_{10}\Big\} \dot{\psi}_{10}\dot{\alpha}_{11} \sin \alpha_{11} \cos \alpha_{11}$$

$$+ 2 (C_{10} + C_{49} - C_{50}) \dot{\psi}_{10}\dot{\alpha}_{11} (\cos^2\alpha_{11}$$

$$- \sin^2\alpha_{11}) T_{33}{}^{10} \sin \theta_{10}$$

$$+ \Big\{C_{47} + \Big[(2 C_{10} + 2 C_{49} - C_{50}) \sin^2\phi_{10}$$

$$+ C_{50} \cos^2\phi_{10}\Big] \sin^2\alpha_{11}$$

$$+ C_{50} \cos^2\alpha_{11}\Big\} \dot{\theta}_{10}\dot{\phi}_{10} \cos \theta_{10}$$

$$+ \Big\{ \Big[(2C_{10} + 2 C_{47} - C_{50}) \sin^2\alpha_{11} + C_{50} \cos^2\alpha_{11}\Big] \sin \theta_{10}$$

$$- 2 (C_{10} + C_{49} - C_{50}) T_{33}{}^{10} \sin \alpha_{11} \cos \alpha_{11}\Big\} \dot{\theta}_{10}\dot{\alpha}_{11} \sin\phi_{10}$$

$$+ \Big\{2 (C_{10} + C_{49} - C_{50}) \sin \theta_{10} \sin \alpha_{11} \cos \alpha_{11}$$

$$- \Big[(2 C_{10} + 2 C_{49} - C_{50}) \cos^2\alpha_{11}$$

$$+ C_{50} \sin^2\alpha_{11}\Big] T_{33}{}^{10}\Big\} \dot{\phi}_{10}\dot{\alpha}_{11}$$

$$+ C_{32} (D_{13}{}^{1}T_{21}{}^{10} - D_{23}{}^{1}T_{11}{}^{10}) - C_{33} (D_{12}{}^{1}T_{21}{}^{10}$$

$$- D_{22}{}^{1}T_{11}{}^{10})$$

$$- C_{34}\ (D_{13}{}^{1}T_{23}{}^{11} - D_{23}{}^{1}T_{13}{}^{11})$$

$$+ C_{35}\ (D_{12}{}^{1}T_{23}{}^{11} - D_{22}{}^{1}T_{13}{}^{11})$$

$$+ C_{36}\ (D_{11}{}^{10}T_{23}{}^{11} - D_{21}{}^{10}T_{13}{}^{11} + D_{13}{}^{11}T_{21}{}^{10} - D_{23}{}^{11}T_{11}{}^{10})$$

$$B\ (26) = (C_{10} + C_{49} - C_{50}) \Big[\dot{\psi}_{10}{}^{2}\ (\sin^2\theta_{10} - \cos^2\theta_{10})$$

$$+ \dot{\phi}_{10}{}^{2} \Big] \cos\phi_{10} \sin\alpha_{11} \cos\alpha_{11}$$

$$- \Big\{ C_9 - C_{47} + C_{48} + (C_{10} + C_{49} \cos^2\phi_{10} - C_{50}) \cos^2\alpha_{11}$$

$$- \Big[(C_{10} - C_{50}) \cos^2\phi_{10} + C_{49} \Big] \sin^2\alpha_{11}$$

$$+ C_{49} \sin^2\phi_{10} \Big\}\ \dot{\psi}_{10}{}^{2} \sin\theta_{10} \cos\theta_{10}$$

$$- \Big\{ C_{47} + \Big[(2\ C_{10} + 2\ C_{49} - C_{50}) \cos^2\phi_{10}$$

$$+ C_{50} \sin^2\phi_{10} \Big] \sin^2\alpha_{11} + C_{50} \cos^2\alpha_{11} \Big\}\ \dot{\psi}_{10}\dot{\phi}_{10} \cos\theta_{10}$$

$$+ 2\ (C_{10} + C_{49} - C_{50})\ (\dot{\phi}_{10}T_{31}{}^{10}$$

$$- \dot{\alpha}_{11}T_{32}{}^{10})\ \dot{\psi}_{10} \cos\phi_{10} \sin\alpha_{11} \cos\alpha_{11}$$

$$- \Big[(2\ C_{10} + 2\ C_{49} - C_{50}) \cos^2\alpha_{11}$$

$$+ C_{50} \sin^2\alpha_{11} \Big]\ \dot{\psi}_{10}\dot{\alpha}_{11} \sin\theta_{10} \sin\phi_{10}$$

$$+ 2\,(C_{10} + C_{49} - C_{50})\,(\dot{\phi}_{10} \cos\phi_{10} \sin\alpha_{11}$$

$$+ \dot{\alpha}_{11} \sin\phi_{10} \cos\alpha_{11})\,\dot{\theta}_{10} \sin\phi_{10} \sin\alpha_{11}$$

$$+ \Big[(2\,C_{10} + 2\,C_{49} - C_{50}) \cos^2\alpha_{11}$$

$$+ C_{50} \sin^2\alpha_{11}\Big]\ \dot{\phi}_{10}\dot{\alpha}_{11} \sin\phi_{10}$$

$$+ (C_{32} \sin\theta_{10} + C_{34}T_{33}{}^{11})\,(D_{13}{}^{1} \cos\psi_{10} + D_{23}{}^{1} \sin\psi_{10})$$

$$+ \Big[C_{32} \cos\theta_{10} + C_{34}\,(\cos\theta_{10} \cos\alpha_{11}$$

$$- \sin\theta_{10} \cos\phi_{10} \sin\alpha_{11})\Big]\ D_{33}{}^{1}$$

$$- (C_{33} \sin\theta_{10} + C_{35}T_{33}{}^{11})\,(D_{12}{}^{1} \cos\psi_{10} + D_{22}{}^{1} \sin\psi_{10})$$

$$- \Big[C_{33} \cos\theta_{10} + C_{35}\,(\cos\theta_{10} \cos\alpha_{11}$$

$$- \sin\theta_{10} \cos\phi_{10} \sin\alpha_{11})\Big]\ D_{32}{}^{1}$$

$$+ C_{36}\ \Big[(D_{13}{}^{11} \cos\psi_{10} + D_{23}{}^{11} \sin\psi_{10}) \sin\theta_{10}$$

$$+ D_{33}{}^{11} \cos\theta_{10}$$

$$- (D_{11}{}^{10} \cos\psi_{10} + D_{21}{}^{10} \sin\psi_{10})\,T_{33}{}^{11}$$

$$- D_{31}{}^{10}\,(\cos\theta_{10} \cos\alpha_{11} - \sin\theta_{10} \cos\phi_{10} \sin\alpha_{11})\Big]$$

$$
\begin{aligned}
B\ (27) = {} & (C_{10} + C_{49} - C_{50})\ \dot{\psi}_{10}{}^{2}T_{32}{}^{10} \sin \theta_{10} \sin \alpha_{11} \cos \alpha_{11} \\
& + (C_{10} + C_{49} - C_{50})\ (\dot{\psi}_{10}{}^{2} \cos^{2}\theta_{10} \\
& - \dot{\theta}_{10}{}^{2}) \sin \phi_{10} \cos \phi_{10} \sin^{2}\alpha_{11} \\
& + 2\ (C_{10} + C_{49} - C_{50})\ (\dot{\theta}_{10} \cos \phi_{10} \\
& + \alpha_{11})\ \dot{\psi}_{10} \sin \theta_{10} \sin \alpha_{11} \cos \alpha_{11} \\
& + \Big\{ C_{47} + \Big[(2\ C_{10} + 2\ C_{49} - C_{50}) \cos^{2}\phi_{10} \\
& + C_{50} \sin^{2}\phi_{10} \Big] \sin^{2}\alpha_{11} + C_{50} \cos^{2}\alpha_{11} \Big\}\ \dot{\psi}_{10}\dot{\theta}_{10} \cos \theta_{10} \\
& + \Big[(2\ C_{10} + 2\ C_{49} - C_{50}) \sin^{2}\alpha_{11} + C_{50} \cos^{2}\alpha_{11} \Big]\ (\dot{\psi}_{10}T_{33}{}^{10} \\
& - \dot{\theta}_{10} \sin \phi_{10})\ \dot{\alpha}_{11} \\
& - 2\ (C_{10} + C_{49} - C_{50})\ \dot{\phi}_{10}\dot{\alpha}_{11} \sin \alpha_{11} \cos \alpha_{11} \\
& - \Big[C_{34}\ (D_{13}{}^{1}T_{12}{}^{10} + D_{23}{}^{1}T_{22}{}^{10} + D_{33}{}^{1}T_{32}{}^{10}) \\
& - C_{35}\ (D_{12}{}^{1}T_{12}{}^{10} + D_{22}{}^{1}T_{22}{}^{10} + D_{32}{}^{1}T_{32}{}^{10}) \\
& - C_{36}\ (D_{11}{}^{10}T_{12}{}^{10} + D_{21}{}^{10}T_{22}{}^{10} \\
& + D_{31}{}^{10}T_{32}{}^{10}) \Big] \sin \alpha_{11}
\end{aligned}
$$

$$B\ (28) = (C_{10} + C_{49} - C_{50})\ \dot{\psi}_{10}^{\ 2}\ (\sin^2\alpha_{11} - \cos^2\alpha_{11})\ T_{33}^{\ 10} \sin\theta_{10}$$

$$+ \Big\{ (C_{10} \cos^2\phi_{10} + C_{49} - C_{50}) \sin^2\theta_{10}$$

$$- \left[C_{10} + (C_{49} - C_{50}) \cos^2\phi_{10}\right] \cos^2\theta_{10}$$

$$+ C_{10} \sin^2\phi_{10} \Big\}\ \dot{\psi}_{10}^{\ 2} \sin\alpha_{11} \cos\alpha_{11}$$

$$+ (C_{10} + C_{49} - C_{50}) \left[\dot{\phi}_{10}^{\ 2} - \dot{\theta}_{10}^{\ 2} \sin^2\phi_{10}\right.$$

$$\left. + 2\dot{\psi}_{10}\ (\dot{\theta}_{10} T_{32}^{\ 10} \cos\phi_{10} - \dot{\phi}_{10} \sin\theta_{10})\right] \sin\alpha_{11} \cos\alpha_{11}$$

$$+ \left[(2C_{10} - 2C_{49} - C_{50}) \cos^2\alpha_{11}\right.$$

$$\left. + C_{50} \sin^2\alpha_{11}\right]\ \dot{\psi}_{10}\dot{\theta}_{10} \sin\theta_{10} \sin\phi_{10}$$

$$+ \left[(2C_{10} + 2C_{49} - C_{50}) \sin^2\alpha_{11} + C_{50} \cos^2\alpha_{11}\right] (\dot{\theta}_{10} \sin\phi_{10}$$

$$- \dot{\psi}_{10} T_{33}^{\ 10})\ \dot{\phi}_{10}$$

$$+ C_{34}\ (D_{13}^{\ 1} T_{11}^{\ 11} + D_{23}^{\ 1} T_{21}^{\ 11} + D_{33}^{\ 1} T_{31}^{\ 11})$$

$$- C_{35}\ (D_{12}^{\ 1} T_{11}^{\ 11} + D_{22}^{\ 1} T_{21}^{\ 11} + D_{32}^{\ 1} T_{31}^{\ 11})$$

$$- C_{36}\ (D_{11}^{\ 10} T_{11}^{\ 11} + D_{21}^{\ 10} T_{21}^{\ 11} + D_{31}^{\ 10} T_{31}^{\ 11})$$

Below are defined the constants $D_{ij}{}^{n}$ used in the B (N) on the previous pages.

$n = 3, 8, 10$

$$D_{11}{}^{n} = (\dot{\psi}_n{}^2 + \dot{\theta}_n{}^2)\ T_{11}{}^{n} - 2\ \dot{\psi}_n \dot{\theta}_n \sin \psi_n \sin \theta_n$$

$$D_{21}{}^{n} = (\dot{\psi}_n{}^2 + \dot{\theta}_n{}^2)\ T_{21}{}^{n} + 2\ \dot{\psi}_n \dot{\theta}_n \cos \psi_n \sin \theta_n$$

$$D_{31}{}^{n} = -\ \dot{\theta}_n{}^2 \sin \theta_n$$

$n = 1,2$

$$D_{12}{}^{n} = (\dot{\psi}_n{}^2 + \dot{\phi}_n{}^2)\ T_{12}{}^{n} + \dot{\theta}_n{}^2 \cos \psi_n \sin \theta_n \sin \phi_n$$

$$+ 2\ \dot{\psi}_n \dot{\theta}_n \sin \psi_n \cos \theta_n \sin \phi_n$$

$$+ 2\ \dot{\psi}_n \dot{\phi}_n\ T_{23}{}^{n}$$

$$- 2\ \dot{\theta}_n \dot{\phi}_n \cos \psi_n \cos \theta_n \cos \phi_n$$

$$D_{22}{}^{n} = (\dot{\psi}_n{}^2 + \dot{\phi}_n{}^2)\ T_{22}{}^{n} + \dot{\theta}_n{}^2 \sin \psi_n \sin \theta_n \sin \phi_n$$

$$- 2\ \dot{\psi}_n \dot{\theta}_n \cos \psi_n \cos \theta_n \sin \phi_n$$

$$- 2\ \dot{\psi}_n \dot{\phi}_n\ T_{13}{}^{n}$$

$$- 2\ \dot{\theta}_n \dot{\phi}_n \sin \psi_n \cos \theta_n \cos \phi_n$$

$$D_{32}{}^{n} = (\dot{\theta}_n{}^2 + \dot{\phi}_n{}^2)\ T_{32}{}^{n} + 2\ \dot{\theta}_n \dot{\phi}_n \sin \theta_n \cos \phi_n$$

$$n = 1, 2, 3, 4, 6$$

$$D_{13}^{n} = (\dot{\psi}_n^{2} + \dot{\phi}_n^{2})\, T_{13}^{n} + \dot{\theta}_n^{2} \cos\psi_n \sin\theta_n \cos\phi_n$$

$$+ 2\,\dot{\psi}_n\dot{\theta}_n \sin\psi_n \cos\theta_n \cos\phi_n$$

$$- 2\,\dot{\psi}_n\dot{\phi}_n\, T_{22}^{n}$$

$$+ 2\,\dot{\theta}_n\dot{\phi}_n \cos\psi_n \cos\theta_n \sin\phi_n$$

$$D_{23}^{n} = (\dot{\psi}_n^{2} + \dot{\phi}_n^{2})\, T_{23}^{n} + \dot{\theta}_n^{2} \sin\psi_n \sin\theta_n \cos\phi_n$$

$$- 2\,\dot{\psi}_n\dot{\theta}_n \cos\psi_n \cos\theta_n \cos\phi_n$$

$$+ 2\,\dot{\psi}_n\dot{\phi}_n\, T_{12}^{n}$$

$$+ 2\,\dot{\theta}_n\dot{\phi}_n \sin\psi_n \cos\theta_n \sin\phi_n$$

$$D_{33}^{n} = (\dot{\theta}_n^{2} + \dot{\phi}_n^{2})\, T_{33}^{n} - 2\,\dot{\theta}_n\dot{\phi}_n \sin\theta_n \sin\phi_n$$

$$\ell = 5,7;\ n = \ell - 1$$

$$D_{11}^{\ell} = (\dot{\psi}_n^{2} + \dot{\alpha}_\ell^{2})\, T_{11}^{\ell}$$

$$+ \dot{\theta}_n^{2} \cos\psi_n\, (\cos\theta_n \sin\alpha_\ell - \sin\theta_n \cos\phi_n \cos\alpha_\ell)$$

$$- \dot{\theta}_n^{2} T_{11}^{n} \cos\alpha_\ell$$

$$+ 2\,\dot{\psi}_n\dot{\theta}_n\, T_{31}^{\ell} \sin\psi_n$$

$$+ 2 \dot{\psi}_n \dot{\phi}_n T_{22}{}^n \cos \alpha_\ell$$

$$+ 2 \dot{\psi}_n \dot{\alpha}_\ell T_{23}{}^\ell$$

$$- 2 \dot{\theta}_n \dot{\phi}_n T_{32}{}^n \cos \psi_n \cos \alpha_\ell$$

$$- 2 \dot{\theta}_n \dot{\phi}_\ell T_{33}{}^\ell \cos \psi_n$$

$$+ 2 \dot{\phi}_n \dot{\alpha} \; T_{12}{}^n \sin \alpha_\ell$$

$$D_{21}{}^\ell = (\dot{\psi}_n{}^2 + \dot{\alpha}_\ell{}^2) T_{21}{}^\ell$$

$$+ \dot{\theta}_n{}^2 \sin \psi_n (\cos \theta_n \sin \alpha_\ell - \sin \theta_n \cos \phi_n \cos \alpha_\ell)$$

$$- \dot{\phi}_n{}^2 T_{23}{}^n \cos \alpha_\ell$$

$$- 2 \dot{\psi}_n \dot{\theta}_n T_{31}{}^\ell \cos \psi_n$$

$$- 2 \dot{\psi}_n \dot{\phi}_n T_{12}{}^n \cos \alpha_\ell$$

$$- 2 \dot{\psi}_n \dot{\alpha}_\ell T_{13}{}^\ell$$

$$- 2 \dot{\theta}_n \dot{\phi}_n T_{32}{}^n \sin \psi_n \cos \alpha_\ell$$

$$- 2 \dot{\theta}_n \dot{\alpha}_\ell T_{33} \sin \psi_n$$

$$+ 2 \dot{\phi}_n \dot{\alpha}_\ell T_{22}{}^n \sin \alpha_\ell$$

$$D_{31}{}^{\ell} = (\dot{\theta}_n{}^2 + \dot{\alpha}_\ell{}^2)\, T_{31}{}^{\ell}$$

$$- \dot{\phi}_n{}^2\, T_{33}{}^{n} \cos \alpha_\ell$$

$$+ 2\, \dot{\theta}_n \dot{\phi}_n \sin \theta_n \sin \phi_n \cos \alpha_\ell$$

$$+ 2\, \dot{\theta}_n \dot{\alpha}_\ell\, (\cos \theta_n \cos \alpha_\ell + \sin \theta_n \cos \phi_n \sin \alpha_\ell)$$

$$+ 2\, \dot{\phi}_n \dot{\alpha}_\ell\, T_{32}{}^{n} \sin \alpha_\ell$$

$$m = 9, 11\ ;\ n = m - 1$$

$$D_{13}{}^{m} = (\dot{\psi}_n{}^2 + \dot{\alpha}_m{}^2)\, T_{13}{}^{m}$$

$$- \dot{\theta}_n{}^2 \cos \psi_n\, (\cos \theta_n \cos \alpha_m - \sin \theta_n \cos \phi_n \sin \alpha_m)$$

$$+ \dot{\phi}_n{}^2\, T_{13}{}^{n} \sin\alpha_m$$

$$+ 2\, \dot{\psi}_n \dot{\theta}_n\, T_{33}{}^{m} \sin \psi_m$$

$$- 2\, \dot{\psi}_n \dot{\phi}_n\, T_{22}{}^{n} \sin \alpha_m$$

$$+ 2\, \dot{\psi}_n \dot{\alpha}_m\, T_{21}{}^{m}$$

$$+ 2\, \dot{\theta}_n \dot{\phi}_n\, T_{32}{}^{n} \cos \psi_n \sin \alpha_m$$

$$- 2\, \dot{\theta}_n \dot{\alpha}_m\, T_{31}{}^{m} \cos \psi_n + 2\, \dot{\phi}_n \dot{\alpha}_m\, T_{12}{}^{n} \cos \alpha_m$$

$$D_{23}{}^{m} = (\dot{\psi}_n{}^2 + \dot{\alpha}_m{}^2)\ T_{23}{}^{m}$$

$$- \dot{\theta}_n{}^2 \sin \psi_n\ (\cos \theta_n \cos \alpha_m - \sin \theta_n \cos \phi_n \sin \alpha_m)$$

$$+ \dot{\phi}_n{}^2\ T_{23}{}^{n} \sin \alpha_m$$

$$- 2\ \dot{\psi}_n \dot{\theta}_n\ T_{33}{}^{m} \cos \psi_n$$

$$+ 2\ \dot{\psi}_n \dot{\phi}_n\ T_{12}{}^{n} \sin \alpha_m$$

$$- 2\ \dot{\psi}_n \dot{\alpha}_m\ T_{11}{}^{m}$$

$$+ 2\ \dot{\theta}_n \dot{\phi}_n\ T_{32}{}^{n} \sin \psi_n \sin \alpha_m$$

$$- 2\ \dot{\theta}_n \dot{\alpha}_m\ T_{31}{}^{m} \sin \psi_n$$

$$+ 2\ \dot{\phi}_n \dot{\alpha}_m\ T_{22}{}^{n} \cos \alpha_m$$

$$D_{33}{}^{m} = (\dot{\theta}_n{}^2 + \dot{\alpha}_m{}^2)\ T_{33}{}^{m}$$

$$+ \dot{\phi}_n{}^2\ T_{33}{}^{n} \sin \alpha_m$$

$$- 2\ \dot{\theta}_n \dot{\phi}_n \sin \theta_n \sin \phi_n \sin \alpha_m$$

$$+ 2\ \dot{\theta}_n \dot{\alpha}_m\ (\cos \theta_n \sin \alpha_m + \sin \theta_n \cos \phi_n \cos \alpha_m)$$

$$+ 2\ \dot{\phi}_n \dot{\alpha}_m\ T_{32}{}^{n} \cos \alpha_m$$

APPENDIX E

ELEMENTS OF VECTOR {P}

$$F_{p_1} = 0$$

$$F_{p_2} = 0$$

$$F_{p_3} = -C_1 g$$

$$F_{p_4} = 0$$

$$F_{p_5} = C_{11} g \sin \theta_1 \cos \phi_1$$

$$F_{p_6} = C_{11} g T_{32}{}^1$$

$$F_{p_7} = 0$$

$$F_{p_8} = C_{12} g \sin \theta_2 \cos \phi_2$$

$$F_{p_9} = C_{12} g T_{32}{}^3$$

$$F_{p_{10}} = 0$$

$$F_{p_{11}} = (C_{13} \sin \theta_3 \cos \phi_3 + C_{14} \cos \theta_3)\, g$$

$$F_{p_{12}} = C_{13} g T_{32}{}^3$$

$$F_{p_{13}} = 0$$

$$F_{p_{14}} = -g \left[C_{15} \sin \theta_4 \cos \phi_4 + C_{16} (\sin \theta_4 \cos \phi_4 \cos \alpha_5 - \cos \theta_4 \sin \alpha_5)\right]$$

$$F_{p_{15}} = -\ (C_{15} + C_{16} \cos \alpha_5)\ gT_{32}^{\ 4}$$

$$F_{p_{16}} = -\ C_{16} gT_{33}^{\ 5}$$

$$F_{p_{17}} = 0$$

$$F_{p_{18}} = -g\ C_{15} \Big[\sin \theta_6 \cos \phi_6 + C_{16} (\sin \theta_6 \cos \phi_6 \cos \alpha_7 - \cos \theta_6 \sin \alpha_7) \Big]$$

$$F_{p_{19}} = -\ (C_{15} + C_{16} \cos \alpha_7)\ gT_{32}^{\ 6}$$

$$F_{p_{20}} = -C_{16} gT_{33}^{\ 7}$$

$$F_{p_{21}} = 0$$

$$F_{p_{22}} = g \Big[C_{17} \cos \theta_8 + C_{18} (\cos \theta_8 \cos \alpha_9 - \sin \theta_8 \cos \phi_8 \sin \alpha_9) \Big]$$

$$F_{p_{23}} = -C_{18} gT_{32}^{\ 8} \sin \alpha_9$$

$$F_{p_{24}} = C_{18} gT_{31}^{\ 9}$$

$$F_{p_{25}} = 0$$

$$F_{p_{26}} = g \Big[C_{17} \cos \theta_{10} + C_{18} (\cos \theta_{10} \cos \alpha_{11} - \sin \theta_{10} \cos \phi_{10} \sin \alpha_{11}) \Big]$$

$$F_{p_{27}} = -C_{18}gT_{32}^{10} \sin \alpha_{11}$$

$$F_{p_{28}} = C_{18}gT_{31}^{11}$$

APPENDIX F

ELEMENTS OF VECTOR {R}

Joint Resistance

Joint Identification

1 : spine, between segments 1 and 2 (between T-12 and L-1)

2 : neck (between C-7 and T-1)

3 : right shoulder

4 : right elbow

5 : left shoulder

6 : left elbow

7 : right hip

8 : right knee

9 : left hip

10 : left knee

Elements of {R}:

$$R_j = - \sum_{i=1}^{10} (M_i + J_i \dot{\beta}_i) \frac{\partial \beta_i}{\partial q_j} \quad (j = 1, 2, \ldots, 28)$$

where $\beta_1 = \cos^{-1} ({}^{1}T_{13} {}^{2}T_{13} + {}^{1}T_{23} {}^{2}T_{23} + {}^{1}T_{33} {}^{2}T_{33})$

$= \cos^{-1} H_1$

H_i for i = 1, 2, 3, 5, 7, and 9 are defined below

$$H_1 = T_{13}^{1}T_{13}^{2} + T_{23}^{1}T_{23}^{2} + T_{33}^{1}T_{33}^{2}$$

$$H_2 = T_{13}^{2}T_{13}^{3} + T_{23}^{2}T_{23}^{3} + T_{33}^{2}T_{33}^{3}$$

$$H_3 = T_{13}^{2}T_{13}^{4} + T_{23}^{2}T_{23}^{4} + T_{33}^{2}T_{33}^{4}$$

$$H_5 = T_{13}^{2}T_{13}^{6} + T_{23}^{2}T_{23}^{6} + T_{33}^{2}T_{33}^{6}$$

$$H_7 = T_{11}^{1}T_{11}^{8} + T_{21}^{1}T_{21}^{8} + T_{31}^{1}T_{31}^{8}$$

$$H_9 = T_{11}^{1}T_{11}^{10} + T_{21}^{1}T_{21}^{10} + T_{31}^{1}T_{31}^{10}$$

whereas, for the elbow and knee joints,

$$\beta_4 = \alpha_5$$
$$\beta_6 = \alpha_7$$
$$\beta_8 = \alpha_9$$
$$\beta_{10} = \alpha_{11}$$

The derivatives required for computation of R_j are

$$\frac{\partial \beta_i}{\partial q_j} = - \frac{1}{\sqrt{1 - H_i^2}} \frac{\partial H_i}{\partial q_j}$$

and

$$\dot{\beta}_i = -\frac{1}{\sqrt{1 - H_i^{\,2}}}\dot{H}_i$$

$$(i = 1, 2, 3, 5, 7, 9,; \; j = 1, 2, \ldots, 28)$$

Defining $G_i = \dot{H}_i$ for $i = 1, 2, 3, 5, 7, 9$

$$G_1 = \dot{T}_{13}{}^{1}T_{13}{}^{2} + \dot{T}_{23}{}^{1}T_{23}{}^{2} + \dot{T}_{33}{}^{1}T_{33}{}^{2}$$

$$+ T_{13}{}^{1}\dot{T}_{13}{}^{2} + T_{23}{}^{1}\dot{T}_{23}{}^{2} + T_{33}{}^{1}\dot{T}_{33}{}^{2}$$

$$G_2 = \dot{T}_{13}{}^{2}T_{13}{}^{3} + \dot{T}_{23}{}^{2}T_{23}{}^{3} + \dot{T}_{33}{}^{2}T_{33}{}^{3}$$

$$+ T_{13}{}^{2}\dot{T}_{13}{}^{3} + T_{23}{}^{2}\dot{T}_{23}{}^{3} + T_{33}{}^{2}\dot{T}_{33}{}^{3}$$

$$G_3 = \dot{T}_{13}{}^{2}T_{13}{}^{4} + \dot{T}_{23}{}^{2}T_{23}{}^{4} + \dot{T}_{33}{}^{2}T_{33}{}^{4}$$

$$+ T_{13}{}^{2}\dot{T}_{13}{}^{4} + T_{23}{}^{3}\dot{T}_{23}{}^{4} + T_{33}{}^{2}\dot{T}_{33}{}^{4}$$

$$G_5 = \dot{T}_{13}{}^{2}T_{13}{}^{6} + \dot{T}_{23}{}^{2}T_{23}{}^{6} + \dot{T}_{33}{}^{2}T_{33}{}^{6}$$

$$+ T_{13}{}^{2}\dot{T}_{13}{}^{6} + T_{23}{}^{2}\dot{T}_{23}{}^{6} + T_{33}{}^{2}\dot{T}_{33}{}^{6}$$

$$G_7 = \dot{T}_{11}{}^{1}T_{11}{}^{8} + \dot{T}_{21}{}^{1}T_{21}{}^{8} + \dot{T}_{31}{}^{1}T_{31}{}^{8}$$

$$+ T_{11}{}^{1}\dot{T}_{11}{}^{8} + T_{21}{}^{1}\dot{T}_{21}{}^{8} + T_{31}{}^{1}\dot{T}_{31}{}^{8}$$

$$G_9 = \dot{T}_{11}{}^{1}T_{11}{}^{10} + \dot{T}_{21}{}^{1}T_{21}{}^{10} + \dot{T}_{31}{}^{1}T_{31}{}^{10}$$

$$+ T_{11}{}^{1}\dot{T}_{11}{}^{10} + T_{21}{}^{1}\dot{T}_{21}{}^{10} + T_{31}{}^{1}\dot{T}_{31}{}^{10}$$

where the time derivatives of the transformation matrix elements are given by

$$\dot{T}_{11}{}^{n} = - \dot{\psi}_n T_{21}{}^{n} + \dot{\theta}_n T_{31}{}^{n} \cos \psi_n$$

$$\dot{T}_{21}{}^{n} = \dot{\psi}_n T_{11}{}^{n} + \dot{\theta}_n T_{31}{}^{n} \sin \psi_n$$

$$\dot{T}_{31}{}^{n} = - \dot{\theta}_n \cos \theta_n$$

$$\dot{T}_{13}{}^{n} = - \dot{\psi}_n T_{23}{}^{n} + \dot{\theta}_n T_{11}{}^{n} \cos \phi_n - \dot{\phi}_n T_{12}{}^{n}$$

$$\dot{T}_{23}{}^{n} = \dot{\psi}_n T_{13}{}^{n} + \dot{\theta}_n T_{21}{}^{n} \cos \phi_n - \dot{\phi}_n T_{22}{}^{n}$$

$$\dot{T}_{33}{}^{n} = \dot{\theta}_n T_{31}{}^{n} \cos \phi_n - \dot{\phi}_n T_{32}{}^{n}$$

The non-zero elements of $\{R\}$ are

$$R_4 = \frac{1}{\sqrt{1 - H_1{}^2}} \left(M_1 - \frac{J_1 \; G_1}{\sqrt{1 - H_1{}^2}}\right) \left(T_{13}{}^1 T_{23}{}^2 - T_{23}{}^1 T_{13}{}^2\right)$$

$$+ \frac{1}{\sqrt{1 - H_7{}^2}} \left(M_1 - \frac{J_7 \; G_7}{\sqrt{1 - H_7{}^2}}\right) \left(T_{11}{}^1 T_{21}{}^8 - T_{21}{}^1 T_{11}{}^8\right)$$

$$+ \frac{1}{\sqrt{1 - H_9{}^2}} \left(M_9 - \frac{J_9 \; G_9}{\sqrt{1 - H_9{}^2}}\right) \left(T_{11}{}^1 T_{21}{}^{10} - T_{21}{}^1 T_{11}{}^{10}\right)$$

$$R_5 = \frac{1}{\sqrt{1 - H_1^{\,2}}} \left(M_1 - \frac{J_1\ G_1}{\sqrt{1 - H_1^{\,2}}}\right) \left(T_{11}^{\ 1} T_{13}^{\ 2} + T_{21}^{\ 1} T_{23}^{\ 2}\right.$$

$$\left. + T_{31}^{\ 1} T_{33}^{\ 2}\right) \cos \phi_1$$

$$- \frac{1}{\sqrt{1 - H_7^{\,2}}} \left(M_7 - \frac{J_7\ G_7}{\sqrt{1 - H_7^{\,2}}}\right) \left[\left(T_{11}^{\ 8} \cos \psi_1\right.\right.$$

$$\left.\left. + T_{21}^{\ 8} \sin \psi_1\right) \sin \theta_1 + T_{31}^{\ 8} \cos \vartheta_1\right]$$

$$- \frac{1}{\sqrt{1 - H_9^{\,2}}} \left(M_9 - \frac{J_9\ G_9}{\sqrt{1 - H_9^{\,2}}}\right) \left[\left(T_{11}^{\ 10} \cos \psi_1\right.\right.$$

$$\left.\left. + T_{21}^{\ 10} \sin \psi_1\right) \sin \theta_1 + T_{31}^{\ 10} \cos \theta_1\right]$$

$$R_6 = - \frac{1}{\sqrt{1 - H_1^{\,2}}} \left(M_1 - \frac{J_1\ G_1}{\sqrt{1 - H_1^{\,2}}}\right) \left(T_{12}^{\ 1} T_{13}^{\ 2} + T_{22}^{\ 1} T_{23}^{\ 2} + T_{32}^{\ 1} T_{33}^{\ 2}\right)$$

$$R_7 = \frac{1}{\sqrt{1 - H_1^{\,2}}} \left(M_1 - \frac{J_1\ G_1}{\sqrt{1 - H_1^{\,2}}}\right) \left(T_{13}^{\ 2} T_{23}^{\ 1} - T_{23}^{\ 2} T_{13}^{\ 1}\right)$$

$$+ \frac{1}{\sqrt{1 - H_1^{\,2}}} \left(M_2 - \frac{J_2\ G_2}{\sqrt{1 - H_2^{\,2}}}\right) \left(T_{13}^{\ 2} T_{23}^{\ 3} - T_{23}^{\ 2} T_{13}^{\ 3}\right)$$

$$+ \frac{1}{\sqrt{1 - H_3^{\,2}}} \left(M_3 - \frac{J_3\ G_3}{\sqrt{1 - H_3^{\,2}}}\right) \left(T_{13}^{\ 2} T_{23}^{\ 4} - T_{23}^{\ 2} T_{13}^{\ 4}\right)$$

$$+ \frac{1}{\sqrt{1 - H_5^{\,2}}} \left(M_5 - \frac{J_5\ G_5}{\sqrt{1 - H_5^{\,2}}}\right) \left(T_{13}^{\ 2} T_{23}^{\ 6} - T_{23}^{\ 2} T_{13}^{\ 6}\right)$$

$$R_8 = \frac{1}{\sqrt{1 - H_1^2}} \left(M_1 - \frac{J_1 G_1}{\sqrt{1 - H_1^2}}\right) \left(T_{11}{}^2 T_{13}{}^1\right.$$

$$\left. + T_{21}{}^2 T_{23}{}^1 + T_{31}{}^2 T_{33}{}^1\right)$$

$$+ \frac{1}{\sqrt{1 - H_2^2}} \left(M_2 - \frac{J_2 G_2}{\sqrt{1 - H_2^2}}\right) \left(T_{11}{}^2 T_{12}{}^3 + T_{21}{}^2 T_{23}{}^3 + T_{31}{}^2 T_{33}{}^3\right)$$

$$+ \frac{1}{\sqrt{1 - H_3^2}} \left(M_3 - \frac{J_3 G_3}{\sqrt{1 - H_3^2}}\right) \left(T_{11}{}^2 T_{13}{}^4 + T_{21}{}^2 T_{23}{}^4 + T_{31}{}^2 T_{33}{}^4\right)$$

$$+ \frac{1}{\sqrt{1 - H_5^2}} \left(M_5 - \frac{J_5 G_5}{\sqrt{1 - H_5^2}}\right) \left(T_{11}{}^2 T_{13}{}^6 + T_{21}{}^2 T_{23}{}^6\right.$$

$$\left. + T_{31}{}^2 T_{33}{}^6\right) \cos \phi_2$$

$$R_9 = - \frac{1}{\sqrt{1 - H_1^2}} \left(M_1 - \frac{J_1 G_1}{\sqrt{1 - H_1^2}}\right) \left(T_{12}{}^2 T_{13}{}^1 + T_{22}{}^2 T_{23}{}^1 + T_{32}{}^3 T_{33}{}^1\right)$$

$$- \frac{1}{\sqrt{1 - H_2^2}} \left(M_2 - \frac{J_2 G_2}{\sqrt{1 - H_3^2}}\right) \left(T_{12}{}^2 T_{13}{}^3 + T_{22}{}^2 T_{23}{}^3 + T_{32}{}^2 T_{33}{}^3\right)$$

$$- \frac{1}{\sqrt{1 - H_3^2}} \left(M_3 - \frac{J_3 G_3}{\sqrt{1 - H_3^2}}\right) \left(T_{12}{}^2 T_{13}{}^4 + T_{22}{}^2 T_{23}{}^4 + T_{32}{}^2 T_{33}{}^4\right)$$

$$- \frac{1}{\sqrt{1 - H_5^2}} \left(M_5 - \frac{J_5 G_5}{\sqrt{1 - H_5^2}}\right) \left(T_{12}{}^2 T_{13}{}^6 + T_{22}{}^2 T_{23}{}^6 + T_{32}{}^2 T_{33}{}^6\right)$$

$$R_{10} = \frac{1}{\sqrt{1 - H_2{}^2}} \left(M_2 - \frac{J_2\ G_2}{\sqrt{1 - H_2{}^2}}\right) \left(T_{13}{}^{3}T_{23}{}^{2} - T_{23}{}^{3}T_{13}{}^{2}\right)$$

$$R_{11} = \frac{1}{\sqrt{1 - H_2{}^2}} \left(M_2 - \frac{J_2\ G_2}{\sqrt{1 - H_2{}^2}}\right) \left(T_{11}{}^{3}T_{13}{}^{2} + T_{21}{}^{3}\ T_{23}{}^{2}\right.$$

$$\left. + T_{31}{}^{3}T_{33}{}^{2}\right) \cos \phi_3$$

$$R_{12} = -\frac{1}{\sqrt{1 - H_2{}^2}} \left(M_2 - \frac{J_2\ G_2}{\sqrt{1 - H_2{}^2}}\right) \left(T_{12}{}^{3}T_{13}{}^{2} + T_{22}{}^{3}T_{23}{}^{2} + T_{32}{}^{3}T_{33}{}^{2}\right)$$

$$R_{13} = \frac{1}{\sqrt{1 - H_3{}^2}} \left(M_3 - \frac{J_3\ G_3}{\sqrt{1 - H_3{}^2}}\right) \left(T_{13}{}^{4}T_{23}{}^{2} - T_{23}{}^{4}T_{13}{}^{2}\right)$$

$$R_{14} = \frac{1}{\sqrt{1 - H_3{}^2}} \left(M_3 - \frac{J_3\ G_3}{\sqrt{1 - H_3{}^2}}\right) \left(T_{11}{}^{4}T_{13}{}^{2} + T_{21}{}^{4}T_{23}{}^{2}\right.$$

$$\left. + T_{31}{}^{4}T_{33}{}^{2}\right) \cos \phi_4$$

$$R_{15} = -\frac{1}{\sqrt{1 - H_3{}^2}} \left(M_3 - \frac{J_3\ G_3}{\sqrt{1 - H_3{}^2}}\right) \left(T_{12}{}^{4}T_{13}{}^{2} + T_{22}{}^{4}T_{23}{}^{2} + T_{32}{}^{4}T_{33}{}^{2}\right)$$

$$R_{16} = -\left(M_4 + J_4\ \dot{\alpha}_5\right)$$

$$R_{17} = \frac{1}{\sqrt{1 - H_5{}^2}} \left(M_5 - \frac{J_5\ G_5}{\sqrt{1 - H_5{}^2}}\right) \left(T_{13}{}^{6}T_{23}{}^{2} - T_{23}{}^{6}T_{13}{}^{2}\right)$$

$$R_{18} = \frac{1}{\sqrt{1 - H_5{}^2}} \left(M_5 - \frac{J_5\ G_5}{\sqrt{1 - H_5{}^2}}\right) \left(T_{11}{}^{6}T_{13}{}^{2} + T_{21}{}^{6}T_{23}{}^{2}\right.$$

$$\left. + T_{31}{}^{6}T_{33}{}^{2}\right) \cos \phi_6$$

$$R_{19} = - \frac{1}{\sqrt{1 - H_5^2}} \left(M_5 - \frac{J_5\ G_5}{\sqrt{1 - H_5^2}}\right) \left(T_{12}\,{}^6T_{13}{}^2 + T_{22}\,{}^6T_{23}{}^2 + T_{32}\,{}^6T_{33}{}^2\right)$$

$$R_{20} = - \left(M_6 + J_6\ \dot{\alpha}_7\right)$$

$$R_{21} = \frac{1}{\sqrt{1 - H_7^2}} \left(M_7 - \frac{J_7\ G_7}{\sqrt{1 - H_7^2}}\right) \left(T_{11}\,{}^8T_{21}{}^1 - T_{21}\,{}^8T_{11}{}^1\right)$$

$$R_{22} = - \frac{1}{\sqrt{1 - H_7^2}} \left(M_7 - \frac{J_7\ G_7}{\sqrt{1 - H_7^2}}\right) \left(T_{11}{}^1 \cos \psi_8\right.$$

$$\left. + T_{21}{}^1 \sin \psi_8) \sin \theta_8 + T_{31}{}^1 \cos \theta_8\right)$$

$$R_{24} = - (M_8 + J_8\ \dot{\alpha}_9)$$

$$R_{25} = \frac{1}{\sqrt{1 - H_9^2}} \left(M_9 - \frac{J_9\ G_9}{\sqrt{1 - H_9^2}}\right) \left(T_{11}\,{}^{10}T_{21}{}^1 - T_{21}\,{}^{10}T_{11}{}^1\right)$$

$$R_{26} = - \frac{1}{\sqrt{1 - H_9^2}} \left(M_9 - \frac{J_9\ G_9}{\sqrt{1 - H_9^2}}\right) \left(T_{11}{}^1 \cos \psi_{10}\right.$$

$$\left. + T_{21}{}^1 \sin \psi_{10}) \sin \theta_{10} + T_{31}{}^1 \cos \theta_{10}\right)$$

$$R_{28} = - \left(M_{10} + J_{10}\ \dot{\alpha}_{11}\right)$$

APPENDIX G

ELEMENTS OF VECTOR {Q}

Non-zero components of {Q}

$$Q_1 = \sum_{i=1}^{11} F_{x_i}$$

$$Q_2 = \sum_{i=1}^{11} F_{y_i}$$

$$Q_3 = \sum_{i=1}^{11} F_{z_i}$$

$$Q_4 = F_{x_1}\left[-\gamma_{x_1} T_{21}^{\ 1} - \gamma_{y_1} T_{22}^{\ 1} - (\gamma_{z_1} - \rho_1) T_{23}^{\ 1}\right]$$

$$+ F_{y_1}\left[\gamma_{x_1} T_{11}^{\ 1} + \gamma_{y_1} T_{12}^{\ 1} + (\gamma_{z_1} - \rho_1) T_{13}^{\ 1}\right]$$

$$+ \bar{\rho}_1 \left[(F_{x_2} + F_{x_3})\ (-T_{23}^{\ 1}) + (F_{y_2} + F_{y_3})\ (T_{13}^{\ 1})\right]$$

$$- \rho_1 \left[(F_{x_8} + F_{x_9} + F_{x_{10}} + F_{x_{11}})\ (-T_{23}^{\ 1})\right.$$

$$\left. + (F_{y_8} + F_{y_9} + F_{y_{10}}\ F_{y_{11}})\ (T_{13}^{\ 1})\right]$$

$$- L_H \left[(F_{x_8} + F_{x_9} - F_{x_{10}} - F_{x_{11}})\ (-T_{22}^{\ 1})\right.$$

$$\left. + (F_{y_8} + F_{y_9} - F_{y_{10}} - F_{y_{11}})\ (T_{12}^{\ 1})\right]$$

$$
\begin{aligned}
Q_5 = {} & F_{x_1} \left[\gamma_{x_1} T_{31}{}^1 + \gamma_{y_1} T_{32}{}^1 + (\gamma_{z_1} - \rho_1) T_{33}{}^1\right] \cos \psi_1 \\
& + F_{y_1} \left[\gamma_{x_1} T_{31}{}^1 + \gamma_{y_1} T_{32}{}^1 + (\gamma_{z_1} - \rho_1) T_{33}{}^1\right] \sin \psi_1 \\
& - F_{z_1} \left[\gamma_{x_1} \cos \theta_1 + \gamma_{y_1} \sin \theta_1 \sin \phi_1\right. \\
& \left. + (\gamma_{z_1} - \rho_1) \sin \theta_1 \cos \phi_1\right] \\
& + \overline{\rho}_1 \left[(F_{x_2} + F_{x_3}) T_{11}{}^1 + (F_{y_2} + F_{y_3}) T_{21}{}^1\right. \\
& \left. + (F_{z_2} + F_{z_3}) T_{31}{}^1\right] \cos \phi_1 \\
& - \rho_1 \left[(F_{x_8} + F_{x_9} + F_{x_{10}} + F_{x_{11}}) T_{11}{}^1\right. \\
& + (F_{y_8} + F_{y_9} + F_{y_{10}} + F_{y_{11}}) T_{21}{}^1 \\
& \left. + (F_{z_8} + F_{z_9} + F_{z_{10}} + F_{z_{11}}) T_{31}{}^1\right] \cos \phi_1 \\
& - L_H \left[(F_{x_8} + F_{x_9} - F_{x_{10}} - F_{x_{11}}) T_{11}{}^1\right. \\
& + (F_{y_8} + F_{y_9} - F_{y_{10}} - F_{y_{11}}) T_{21}{}^1 \\
& \left. + (F_{z_8} + F_{z_9} - F_{z_{10}} - F_{z_{11}}) T_{31}{}^1\right] \sin \phi_1
\end{aligned}
$$

$$
\begin{aligned}
Q_6 = {} & F_{x_1} \left[\gamma_{y_1} T_{13}{}^1 - (\gamma_{z_1} - \rho_1)\, T_{12}{}^1 \right] \\
& + F_{y_1} \left[\gamma_{y_1} T_{31}{}^1 - (\gamma_{z_1} - \rho_1)\, T_{22}{}^1 \right] \\
& + F_{z_1} \left[\gamma_{y_1} T_{33}{}^1 - (\gamma_{z_1} - \rho_1)\, T_{32}{}^1 \right] \\
& + \bar{\rho}_1 \left[- (F_{x_2} + F_{x_3})\, T_{12}{}^1 - (F_{y_2} + F_{y_3})\, T_{22}{}^1 \right. \\
& \left. - (F_{z_2} + F_{z_3})\, T_{32}{}^1 \right] \\
& + \rho_1 \left[(F_{x_8} + F_{x_9} + F_{x_{10}} + T_{x_{11}})\, T_{12}{}^1 \right. \\
& + (F_{y_8} + F_{y_9} + F_{y_{10}} + F_{y_{11}})\, T_{22}{}^1 \\
& \left. + (F_{z_8} + F_{z_9} + F_{z_{10}} + F_{z_{11}})\, T_{32}{}^1 \right] \\
& - L_H \left[(F_{x_8} + F_{x_9} - F_{x_{10}} - F_{x_{11}})\, T_{13}{}^1 \right. \\
& + (F_{y_8} + F_{y_9} - F_{y_{10}} - F_{y_{11}})\, T_{23}{}^1 \\
& \left. + (F_{z_8} + F_{z_9} - F_{z_{10}} - F_{z_{11}})\, T_{33}{}^1 \right]
\end{aligned}
$$

$$Q_7 = F_{x_2} \left[- \gamma_{x_2} T_{21}^{\ 2} - \gamma_{y_2} T_{22}^{\ 2} - (\rho_2 + \gamma_{z_2}) T_{23}^{\ 2} \right]$$

$$+ F_{y_2} \left[\gamma_{x_2} T_{11}^{\ 2} + \gamma_{y_2} T_{12}^{\ 2} + (\rho_2 + \gamma_{z_2}) T_{13}^{\ 2} \right]$$

$$+ F_{x_3} L_2 (- T_{23}^{\ 2}) + F_{y_3} L_2 (T_{13}^{\ 2})$$

$$Q_8 = F_{x_2} \left[\gamma_{x_2} T_{31}^{\ 2} + \gamma_{y_2} T_{32}^{\ 2} + (\rho_2 + \gamma_{z_2}) T_{33}^{\ 2} \right] \cos \psi_2$$

$$+ F_{y_2} \left[\gamma_{x_2} T_{31}^{\ 2} + \gamma_{y_2} T_{32}^{\ 2} + (\rho_2 - \gamma_{z_2}) T_{33}^{\ 2} \right] \sin \psi_2$$

$$- F_{z_2} \left[\gamma_{x_2} \cos \theta_2 + \gamma_{y_2} \sin \theta_2 \sin \phi_2 \right.$$

$$\left. + (\rho_2 + \gamma_{z_2}) \sin \theta_2 \cos \phi_2 \right]$$

$$+ L_2 \left[F_{x_3} T_{11}^{\ 2} + F_{y_3} T_{21}^{\ 2} + F_{z_3} T_{31}^{\ 2} \right] \cos \phi_2$$

$$Q_9 = F_{x_2} \left[\gamma_{y_2} T_{13}^{\ 2} - (\rho_2 + \gamma_{z_2}) T_{12}^{\ 2} \right]$$

$$+ F_{y_2} \left[\gamma_{y_2} T_{23}^{\ 2} - (\rho_2 + \gamma_{z_2}) T_{22}^{\ 2} \right]$$

$$+ F_{z_2} \left[\gamma_{y_2} T_{33}^{\ 2} - (\rho_2 + \gamma_{z_2}) T_{32}^{\ 2} \right]$$

$$- L_2 \left[F_{x_3} T_{12}^{\ 2} + F_{y_3} T_{22}^{\ 2} + F_{z_3} T_{32}^{\ 2} \right]$$

$$Q_{10} = F_{x_3}\left[-\,(\gamma_{x_3} + L_c)\ T_{21}^{\ 3} - \gamma_{y_3}\ T_{22}^{\ 4} - (\gamma_{z_3} + \rho_3)\ T_{23}^{\ 3}\right]$$

$$+\ F_{y_3}\left[(\gamma_{x_3} + L_c)\ T_{11}^{\ 3} + \gamma_{y_3}\ T_{12}^{\ 3} + (\gamma_{z_3} + \rho_3)\ T_{13}^{\ 3}\right]$$

$$Q_{11} = F_{x_3}\left[(\gamma_{x_3} + L_c)\ T_{31}^{\ 3} + \gamma_{y_3}\ T_{32}^{\ 3} + (\gamma_{z_3} + \rho_3)\ T_{33}^{\ 3}\right] \cos \psi_3$$

$$+\ F_{y_3}\left[(\gamma_{x_3} + L_c)\ T_{31}^{\ 3} + \gamma_{y_3}\ T_{32}^{\ 3} + (\gamma_{z_3} + \rho_3)\ T_{33}^{\ 3}\right] \sin \psi_3$$

$$-\ F_{z_3}\left[(\gamma_{x_3} + L_c) \cos \theta_3 + \gamma_{y_3} \sin \theta_3 \sin \phi_3\right.$$

$$\left. +\ (\gamma_{z_3} + \rho_3) \sin \theta_3 \cos \phi_3\right]$$

$$Q_{12} = F_{x_3}\left[\gamma_{y_3}\ T_{13}^{\ 3} - (\gamma_{z_3} + \rho_3)\ T_{12}^{\ 3}\right]$$

$$+\ F_{y_3}\left[\gamma_{y_3}\ T_{23}^{\ 3} - (\gamma_{z_3} + \rho_3)\ T_{22}^{\ 3}\right]$$

$$+\ F_{z_3}\left[\gamma_{y_3}\ T_{33}^{\ 3} - (\gamma_{z_3} + \rho_3)\ T_{32}^{\ 3}\right]$$

$$Q_{21} = F_{x_8}\left[-\,(\gamma_{x_8} + \rho_8)\ T_{21}^{\ 8} - \gamma_{y_8}\ T_{22}^{\ 8} - \gamma_{z_8}\ T_{23}^{\ 8}\right]$$

$$+\ F_{y_8}\left[(\gamma_{x_8} + \rho_8)\ T_{11}^{\ 8} + \gamma_{y_8}\ T_{12}^{\ 8} + \gamma_{z_8}\ T_{13}^{\ 8}\right]$$

$$+\ F_{x_9}\left[-\,L_8\ T_{21}^{\ 8} - \gamma_{x_9}\ T_{21}^{\ 9} - \gamma_{y_9}\ T_{22}^{\ 8} - (\gamma_{z_9} - L_9)\ T_{23}^{\ 9}\right]$$

$$+\ F_{y_9}\left[L_8\ T_{11}^{\ 8} + \gamma_{x_9}\ T_{11}^{\ 9} + \gamma_{y_9}\ T_{12}^{\ 8} + (\gamma_{z_9} - L_9)\ T_{13}^{\ 9}\right]$$

$$Q_{22} = F_{x_8} \left[(\gamma_{x_8} + \rho_8)\ T_{31}{}^8 + \gamma_{y_8}\ T_{32}{}^8 + \gamma_{z_8}\ T_{33}{}^8\right] \cos \psi_8$$

$$+ F_{y_8} \left[(\gamma_{x_8} + \rho_8)\ T_{31}{}^8 + \gamma_{y_8}\ T_{32}{}^8 + \gamma_{z_8}\ T_{33}{}^8\right] \sin \psi_8$$

$$- F_{z_8} \left[(\gamma_{x_8} + \rho_8) \cos \theta_8 + \gamma_{y_8} \sin \theta_8 \sin \phi_8\right.$$

$$\left. + \gamma_{z_8} \sin \theta_8 \cos \phi_8\right]$$

$$+ F_{x_9} \left[L_8\ T_{31}{}^8 + \gamma_{x_9}\ T_{31}{}^9 + \gamma_{y_9}\ T_{32}{}^8\right.$$

$$\left. + (\gamma_{z_9} - L_9)\ T_{33}{}^9\right] \cos \psi_8$$

$$+ F_{y_9} \left[L_8\ T_{31}{}^8 + \gamma_{x_9}\ T_{31}{}^9 + \gamma_{y_9}\ T_{32}{}^8\right.$$

$$\left. + (\gamma_{z_9} - L_9)\ T_{33}{}^9\right] \sin \psi_8$$

$$- F_{z_9} \left[L_8 \cos \theta_8 + \gamma_{x_9} (\cos \theta_8 \sin \alpha_9 + \sin\theta_8 \cos \phi_8 \cos \alpha_9)\right.$$

$$+ \gamma_{y_9} \sin \theta_8 \sin \phi_8 + (\gamma_{z_9} - L_9) (\sin \theta_8 \cos \phi_8 \sin \alpha_9$$

$$\left. - \cos \theta_8 \cos \alpha_9)\right]$$

$$Q_{23} = F_{x_8} (\gamma_{y_8}\ T_{13}{}^8 - \gamma_{z_8}\ T_{12}{}^8)$$

$$+ F_{y_8} (\gamma_{y_8}\ T_{23}{}^8 - \gamma_{z_8}\ T_{22}{}^8)$$

$$+ F_{z_8} (\gamma_{y_8} T_{33}{}^8 - \gamma_{z_8} T_{32}{}^8)$$

$$+ F_{x_9} \left[- \gamma_{x_9} T_{12}{}^8 \cos \alpha_9 + \gamma_{y_9} T_{13}{}^8 - (\gamma_{z_9} - L_9) T_{12}{}^8 \sin \alpha_9\right]$$

$$+ F_{y_9} \left[-\gamma_{x_9} T_{22}{}^8 \cos \alpha_9 + \gamma_{y_9} T_{23}{}^8 - (\gamma_{z_9} - L_9) T_{22}{}^8 \sin \alpha_9\right]$$

$$+ F_{z_9} \left[-\gamma_{x_9} T_{32}{}^8 \cos \alpha_9 + \gamma_{y_9} T_{33}{}^8 - (\gamma_{z_9} - L_9) T_{32}{}^8 \sin \alpha_9\right]$$

$$Q_{24} = F_{x_9} \left[- \gamma_{x_9} T_{13}{}^8 + (\gamma_{z_9} - L_9) T_{11}{}^9\right]$$

$$+ F_{y_9} \left[-\gamma_{x_9} T_{33}{}^9 + (\gamma_{z_9} - L_9) T_{31}{}^9\right]$$

$$+ F_{z_9} \left[-\gamma_{x_9} T_{33}{}^9 + (\gamma_{z_9} - L_9) T_{31}{}^9\right]$$

$$Q_{25} = - F_{x_{10}} \left[(\gamma_{x_{10}} + \rho_8) T_{21}{}^{10} + \gamma_{y_{10}} T_{22}{}^{10} + \gamma_{z_{10}} T_{23}{}^{10}\right]$$

$$+ F_{y_{10}} \left[(\gamma_{x_{10}} + \rho_8) T_{11}{}^{10} + \gamma_{y_{10}} T_{12}{}^{10} + \gamma_{z_{10}} T_{13}{}^{10}\right]$$

$$- F_{x_{11}} \left[L_8 T_{21}{}^{10} + \gamma_{x_{11}} T_{21}{}^{11} + \gamma_{y_{11}} T_{22}{}^{10} + (\gamma_{z_{11}} - L_9) T_{23}{}^{11}\right]$$

$$+ F_{y_{11}} \left[L_8 T_{11}{}^{10} + \gamma_{x_{11}} T_{11}{}^{11} + \gamma_{y_{11}} T_{12}{}^{10}\right.$$

$$\left. + (\gamma_{z_{11}} - L_9) T_{13}{}^{11}\right]$$

$$Q_{26} = F_{x_{10}} \left[(\gamma_{x_{10}} + \rho_8)\ T_{31}^{10} + \gamma_{y_{10}}\ T_{32}^{10} + \gamma_{z_{10}}\ T_{33}^{10}\right] \cos \psi_{10}$$

$$+ F_{y_{10}} \left[(\gamma_{x_{10}} + \rho_8)\ T_{31}^{10} + \gamma_{y_{10}}\ T_{32}^{10} + \gamma_{z_{10}}\ T_{33}^{10}\right] \sin \psi_{10}$$

$$- F_{z_{10}} \left[(\gamma_{x_{10}} + \rho_8) \cos \theta_{10} + \gamma_{y_{10}} \sin \theta_{10} \sin \phi_{10}\right.$$

$$\left. + \gamma_{z_{10}} \sin \theta_{10} \cos \phi_{10}\right]$$

$$+ (F_{x_{11}} \cos \psi_{10} + F_{y_{11}} \sin \psi_{10}) \left[L_8\ T_{31}^{10} + \gamma_{x_{11}}\ T_{31}^{11}\right.$$

$$\left. + \gamma_{y_{11}} T_{32}^{10} + (\gamma_{z_{11}} - L_9)\ T_{33}^{11}\right]$$

$$- F_{z_{11}} \left[L_8 \cos \theta_{10} + \gamma_{x_{11}} (\cos \theta_{10} \sin \alpha_{11}\right.$$

$$+ \sin \theta_{10} \cos \phi_{10} \cos \alpha_{11}) + \gamma_{y_{11}} \sin \theta_{10} \sin \phi_{10}$$

$$\left. + (\gamma_{z_{11}} - L_9) (\sin \theta_{10} \cos \phi_{10} \sin \alpha_{11} - \cos \theta_{10} \cos \alpha_{11})\right]$$

$$Q_{27} = F_{x_{10}} (\gamma_{y_{10}}\ T_{13}^{10} - \gamma_{z_{10}}\ T_{12}^{10})$$

$$+ F_{y_{10}} (\gamma_{y_{10}}\ T_{23}^{10} - \gamma_{z_{10}}\ T_{22}^{10})$$

$$+ F_{z_{10}} (\gamma_{y_{10}}\ T_{33}^{10} - \gamma_{z_{10}}\ T_{32}^{10})$$

$$+ F_{x_{11}} \left[\gamma_{y_{11}}\ T_{13}^{10} - (\gamma_{x_{11}} \cos \alpha_{11}\right.$$

$$+ \{\gamma_{z_{11}} - L_9\} \sin \alpha_{11}) \, T_{12}^{10}\Big]$$

$$+ F_{y_{11}} \Big[\gamma_{y_{11}} T_{23}^{10} - (\gamma_{x_{11}} \cos \alpha_{11}$$

$$+ \{\gamma_{z_{11}} - L_9\} \sin \alpha_{11}) \, T_{22}^{10}\Big]$$

$$+ F_{z_{11}} \Big[\gamma_{y_{11}} T_{33}^{10} - (\gamma_{x_{11}} \cos \alpha_{11}$$

$$+ \{\gamma_{z_{11}} - L_9\} \sin \alpha_{11}) \, T_{32}^{10}\Big]$$

$$Q_{28} = F_{x_{11}} \Big[-\gamma_{x_{11}} T_{13}^{11} + (\gamma_{z_{11}} - L_9) \, T_{11}^{11}\Big]$$

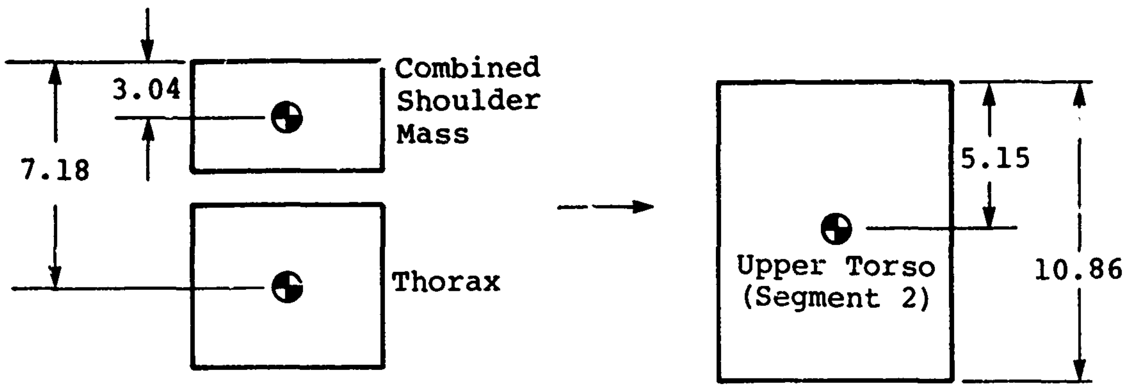
$$+ F_{y_{11}} \Big[-\gamma_{x_{11}} T_{23}^{11} + (\gamma_{z_{11}} - L_9) \, T_{21}^{11}\Big]$$

$$+ F_{z_{11}} \Big[-\gamma_{x_{11}} T_{33}^{1} + (\gamma_{z_{11}} - L_9) \, T_{31}^{11}\Big]$$

APPENDIX H

SEGMENT MOMENT OF INERTIA CALCULATIONS

H.1 CALCULATION OF MOMENT OF INERTIA FOR SEGMENT 2



For Dempster's segments, idealized above (a) the combined mass of the shoulder segments, $m_s = 0.1054$ M where M is total body mass. The mass of the thorax segment, $m_t = 0.1097$ M. For the 50th percentile male with total body weight = 161.5 lb,

$$m_s = 0.1054\ (161.5/386.4) = 0.0441\ \text{lb-sec}^2/\text{in}$$

$$m_t - 0.1097\ (161.5/386.4) = 0.0459\ \text{lb-sec}^2/\text{in}$$

Dempster's moments of inertia for transverse axes passing through the mass centers, adjusted for weight of 161.5 lb are

$$I_s = 2(0.5613) = 1.1226\ \text{lb-sec}^2\text{-in}$$

$$I_t = 1.1149\ \text{lb-sec}^2\text{-in}$$

Using the parallel-axis theorem,

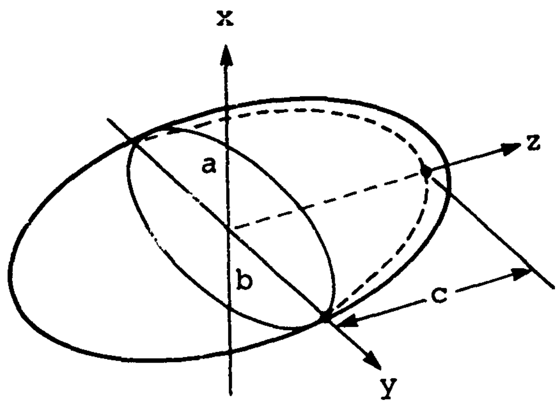
$$I_{y_2} = 1.1226 + 0.0441\ (5.15 - 3.04)^2$$

$$+ 1.1149 + 0.0459\ (7.18 - 5.15)^2$$

$$= 2.623\ \text{lb-sec}^2\text{-in}$$

H.2 CALCULATION OF I_{x_n} AND I_{z_n}

1. Approximate segments 1, 2, and 3 by ellipsoids

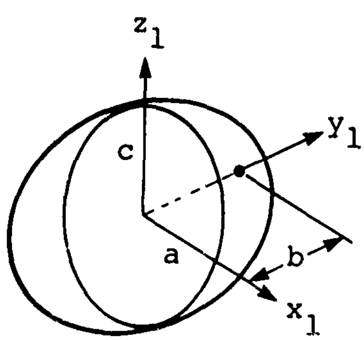


$$I_x = \frac{m}{5} (b^2 + c^2)$$

$$I_y = \frac{m}{5} (a^2 + c^2)$$

$$I_z = \frac{m}{5} (a^2 + b^2)$$

<u>Segment 1</u>



$I_{y_1} = 4.331$ lb-sec^2-in

Assume

$a = c = R_1$ (buttock radius)

$b = L_H + R_{16}$ (1/2 hip breadth)

For 50th percentile

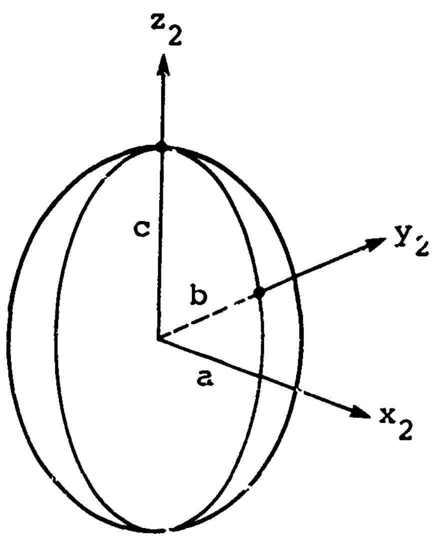
$a = c = 4.0$, $b = 13.9/2$

$$I_{x_1} = \frac{m}{5} \left[\left(\frac{13.9}{2}\right)^2 + (4)^2 \right] = 64.3\ m/5 = I_{z_1}$$

$$I_{y_1} = \frac{m}{5} \left[(4)^2 + (4)^2 \right] = 32.0\ m/5$$

$$I_{x_1} = I_{z_1} = 2.009\ I_{y_1} = 8.703 \text{ lb-sec}^2\text{-in}$$

Segment 2



$I_{y_2} = 2.623$ lb-sec^2-in

$a = R_2$ (chest depth)

$b = c = 1/2$ chest breadth

For 50th percentile

$a = 4.5,\ b = c = 6.0$

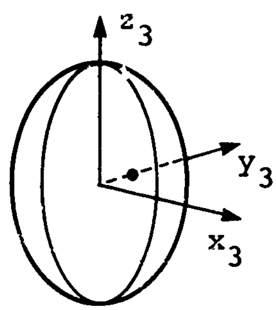
$$I_{x_2} = \frac{m}{5}\left[(6)^2 + (6)^2\right] = 72\ m/5$$

$$I_{y_2} = \frac{m}{5}\left[(4.5)^2 + (6)^2\right] = 56.3\ m/5$$

$$I_{x_2} = 1.28\ I_{y_2} = 3.354 \text{ lb-sec}^2\text{-in}$$

$$I_{y_2} = I_{y_2} = 2.623 \text{ lb-sec}^2\text{-in}$$

Segment 3



$I_{y_3} = 0.311$ lb-sec^2-in

$a = R_3 = 4.0$ in $= b$

$c = \bar{\rho}_3 = 5.8$ in

$$I_{x_3} = \frac{m}{5}\left[(4)^2 + (5.8)^2\right] = 49.6\ m/5$$

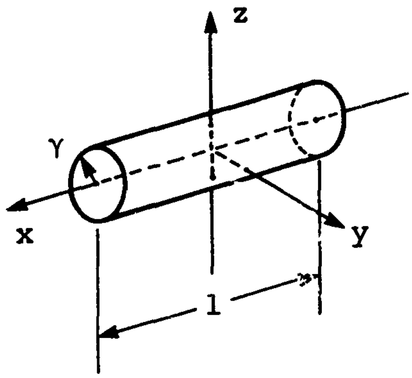
$$I_{y_3} = I_{x_3}$$

$$I_{z_3} = \frac{m}{5}\left[(4)^2 + (4)^2\right] = 32\ m/5$$

$$I_{x_3} = 0.311 \text{ lb-sec}^2\text{-in}$$

$$I_{z_3} = 0.6446\ I_{y_3} = 0.2006 \text{ lb-sec}^2\text{-in}$$

2. Approximate limb segments (4-11) by circular cylinders



$I_x = 1/2\ m\gamma^2$ (assumed to be long axis)

$I_y = I_z = 1/4\ m\gamma^2 + 1/12\ \ell^2$

Upper Arms

$$I_{y_4} = 0.164 \text{ lb-sec}^2\text{-in} = I_{x_4}$$

$$\gamma = R_4 = 3.9/2 \;;\; \ell = L_4 = 11.99$$

$$I_{y_4} = 1/4\ mR_4{}^2 + 1/12\ mL_4{}^2$$

$$I_{z_4} = 1/2\ mR_4{}^2 = 0.0241 \text{ lb-sec}^2\text{-in}$$

$$I_{x_6} = I_{y_6} = I_{x_4} = I_{y_4}$$

$$I_{z_6} = I_{z_4}$$

Forearm and Hand Segments

$$I_{y_5} = 0.218 \text{ lb-sec}^2\text{-in} = I_{z_5}$$

$$\gamma = R_5 \; 1.85 \;;\; \ell = L_5 = 13.23$$

$$I_{y_5} = I_{z_5} = 1/4 \; mR_5{}^2 + 1/12 \; mL_5{}^2$$

$$I_{x_5} = 1/2 \; mR_5{}^2 = 0.0241 \text{ lb-sec}^2\text{-in}$$

$$I_{x_7} = I_{x_5}$$

$$I_{y_7} = I_{z_7} = I_{y_5} = I_{z_5}$$

Thighs

$$I_{y_8} = 1.270 \text{ lb-sec}^2\text{-in} = I_{z_8}$$

$$\gamma = R_8 = 3.55 \;;\; \ell = L_8 = 16.58$$

$$I_{y_8} = I_{z_8} = 1/4 \; mR_8{}^2 + 1/12 \; mL_8{}^2$$

$$I_{x_8} = 1/2 \; mR_8{}^2$$

$$= 0.307 \text{ lb-sec}^2\text{-in}$$

$$I_{x_{10}} = I_{x_8}$$

$$I_{y_{10}} = I_{z_{10}} = I_{y_8} = I_{z_8}$$

Leg and Foot Segments

$$I_{y_9} = 1.192 \text{ lb-sec}^2\text{-in}$$

$$\gamma = R_9 = 2.3 \ ; \ \ell = L_9 = 17.31$$

$$I_{y_9} = 1/4 \ mR_9{}^2 + 1/12 \ mL_9{}^2 = I_{x_9}$$

$$I_{z_9} = 1/2 \ mR_9{}^2$$

$$= 0.1199 \text{ lb-sec}^2\text{-in}$$

$$I_{x_{11}} = I_{y_{11}} = I_{x_9} = I_{y_9}$$

$$I_{z_{11}} = I_{z_9}$$